ON DETERMINISTIC PERTURBATIONS OF SUMMABILITY MAPS

GAO BING

(B.Sc., USTC, China)

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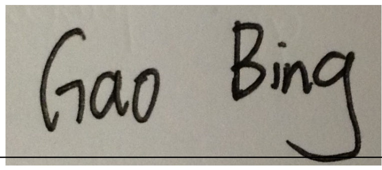
To my parents

DECLARATION

I hereby declare that the thesis is my original work and it has been written by me in its entirety.

I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.



Gao Bing
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Contents

Summary

This thesis contains two topics on perturbations of non-uniformly expanding interval maps.

The first topic is to provide a strengthened version of the famous Jakobson's theorem. Consider an interval map f satisfying a summability condition. For a generic one-parameter family f_t of maps with $f_0 = f$, we prove that $t = 0$ is a Lebesgue density point of the set of parameters for which f_t satisfies both the Collet-Eckmann condition and a strong polynomial recurrence condition.

The second topic is to investigate the asymptotic distributions of the critical orbits. Consider a one-parameter family with some conditions and let Δ be the set of parameters t for which f_t satisfies a summability condition and a transversality condition. We prove that for almost all $t \in \Delta$, each critical point of f_t belongs to the basin of one of the ergodic absolutely continuous invariant probability measure for f_t.

List of notations

- $\mathbb{N} = \{0, 1, 2 \ldots,\}$ denote the set of natural numbers.
- $\mathbb{R}$ denote the set of real numbers.
- $\text{int}(I)$ denote the interior of I.
- $\overline{I}$ denote the closure of I.
- $C(I, J)$ denote the collection of continuous functions from I to J.
- $C^r(I, J)$ denote the collection of C^r maps from I to J.
- $|X|$ denote the Lebesgue measure of $X \subset R$.
- $\text{Leb}_n(X)$ denote the Lebesgue measure of $X \subset \mathbb{R}^n$.

Chapter 1

Introduction

A dynamical system is a rule for time evolution on a state space. Examples and applications arise from all branches of science and technology, like physics, chemistry, economics, etc. In broad terms, one of the main goals of dynamical systems is to describe the typical behavior of orbits for a typical dynamical system. There can be different points of view on the meaning of typical but we are particularly interested in the notion from a probabilistic point of view which makes the best physical sense, see [27]. Roughly speaking, the goal is the following: Given a finite dimensional manifold M and a finite parameter family $f_t : M \to M$ of dynamical systems on M, describe the asymptotic behavior of Lebesgue almost all orbits of f_t for Lebesgue almost all parameters t.

This problem is quite hard in general. It turns out that the one-dimensional dynamical systems, as models for dynamical behavior in high dimensions, deserve a great deal of attention. In this thesis, we focus on real smooth interval maps with finitely many critical points (multimodal maps) exhibiting complicated behavior.

1.1 Studies on stochastic behavior

A differentiable interval map f satisfies *Axiom A*, if the following hold:

- all periodic points are hyperbolic;
- the complement Ω of the basins of periodic attractors is a hyperbolic set for f, that is, there are constants $C > 0$ and $\lambda > 1$ such that $|Df^n(x)| > C\lambda^n$ holds for all $x \in \Omega$ and $n \in \mathbb{N}$.

The dynamics of Axiom A maps is quite well understood. In fact, it is easy to show that the set Ω above is a nowhere dense compact set with zero Lebesgue measure, provided that f is C^2. For such f, Lebesgue almost all orbits converge to some periodic attractor. Moreover, it was shown that any real polynomial can be approximated by real Axiom A polynomials of the same degree, see [17].

An interval map f which does not satisfy Axiom A, may produce extreme dynamical complexity. A way of dealing with the complexity is to introduce an invariant measure μ for f. When μ is ergodic, the Birkhoff ergodic theorem states that for any real valued continuous function ϕ, the time and space average agree, i.e.,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(x)) = \int \phi d\mu \tag{1.1}$$

for μ-a.e. x. This provides us with a good statistical description of some orbits. We are interested in the information which is observable in the physical sense. For this reason, it is natural to require that μ is absolutely continuous with respect to Lebesgue measure.

An interval map f is called *stochastic*, if it has an invariant probability measure μ which is absolutely continuous with respect to Lebesgue measure (abbreviated acip). If μ is ergodic, there exists a positive Lebesgue measure set $B(\mu)$, called *basin of* μ, such that (1.1) holds for any $x \in B(\mu)$ and any real valued continuous function ϕ. So in this case, one can give the predictions about averages.

The famous result of Jakobson [15] states that maps with stochastic behavior are abundant, in the probabilistic sense, in the real quadratic family. This implies that stochastic phenomena can not be neglected in the real quadratic family. This remarkable result opened the way to much progress in non-uniformly expanding dynamics.

It was later realized that sufficient expansion along the orbits of critical values often implies stochastic behavior. In [13], the Collet-Eckmann condition, which requires that for each critical point c, the derivative $|Df^n(f(c))|$ grows exponentially fast with n, guarantees the existence of an acip for S-unimodal maps. In fact, in [13] another, additional assumption was made on the expansion along the backward orbit of critical points, but Nowicki showed that Collet-Eckmann condition implies the backward one. Alternative approach to Jakobson's theorem in [6] and [7] focused on this property: the set of Collet-Eckmann maps in the real quadratic family, has positive Lebesgue measure. A similar result and a more precise estimate for multimodal maps are provided by Tsujii, see [36].

After these works, the maps satisfying the Collet-Eckmann condition, attracted a lot of interest. Many researchers studied Collet-Eckmann parameters, obtaining refined information of Collet-Eckmann maps. In [5], the authors proved that: for a typical stochastic unimodal map, the critical point belongs to the basin of an acip. In other words, for typical stochastic unimodal maps, the critical point is typical for the measure of the system. This is a generalization of the result given by Benedicks and Carleson in [6], who proved typicality of the critical orbit for a positive measure set of parameters for the quadratic family.

1.2 Statements of results

In [26], the summability condition was shown to imply the existence of an acip for S-unimodal maps. In the recent work [11], existence of acip for multimodal map was proved under the large derivatives condition. With these works, it is natural to investigate non-uniformly expanding maps with some weak conditions.

1.2.1 Notations

To state our results, we start with some definitions. Suppose $f \in C^1([0,1],[0,1])$ and let $\mathcal{C}(f)$ denote the set of critical points of f. We say that f satisfies *the summability condition* (abbreviated (SC)), if for each $c \in \mathcal{C}(f)$, we have

$$\sum_{n=0}^{\infty} \frac{1}{|Df^n(f(c))|} < \infty.$$

We say f satisfies *the Collet-Eckmann condition* (abbreviated (CE)), if for each $c \in \mathcal{C}(f)$, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log |Df^n(f(c))| > 0.$$

We say f satisfies *the weak regularity condition* (abbreviated (WR)), if for each $c \in \mathcal{C}(f)$, we have

$$\lim_{\delta\to 0^+} \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(|Df(f^{i+1}(c))|) \cdot 1_{C(f,\delta)}(f^{i+1}(c)) = 0,$$

where $1_{C(f,\delta)}$ is the indicator function of the set $\{x \in [0,1] : \mathrm{dist}(x, \mathcal{C}(f)) < \delta\}$. Furthermore, we say f satisfies the *polynomial recurrence condition of exponent* β (abbreviated (PR_β)), if there exists $C > 0$ such that for any $c, c' \in \mathcal{C}(f)$ and any $n \geq 1$, we have

$$\mathrm{dist}(f^n(c), c') \geq Cn^{-\beta}.$$

If for each $\beta > 1$, f satisfies PR_β, then we say that f satisfies the *strong polynomial recurrence condition* (abbreviated (SPR)).

Let $\mathcal{A}$ be the subset of $C^1([0,1],[0,1])$ with the following properties:

- f has no attracting or neutral periodic orbits;
- each critical point of f lies in the interior $(0,1)$;
- f is C^3 outside $\mathcal{C}(f)$;
- for each critical point c, there exist $\ell > 1$ and a C^3 diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(c) = 0$ and such that $|f(x) - f(c)| = |\varphi(x) - \varphi(c)|^\ell$ holds near c.

For a one parameter C^1 family $F(x,t) = f_t(x)$, we say that f_t satisfies *the (NV_t) condition*, if

$$(\mathrm{NV}_t) \qquad \sum_{j=0}^{\infty} \frac{\partial_t F(f_t^j(c), t)}{Df_t^j(f_t(c))} \neq 0 \text{ for any critical point } c \in \mathcal{C}(f_t)$$

Consider a one-parameter C^1 family $f_t : [0,1] \to [0,1]$, $t \in [-\delta, \delta]$. We say that this family is *regular* if the following hold:

1. The map $(t,x) \mapsto f_t(x)$ is C^2 on $\{(t,x) \in [-\delta,\delta] \times [0,1] : Df_t(x) \neq 0\}$.
2. There exist C^2 functions $c_i : [-\delta,\delta] \to (0,1)$, $i = 1, 2, \ldots, d$, such that $0 < c_1(t) < c_2(t) < \ldots < c_d(t) < 1$ and $\mathcal{C}(f_t) = \{c_i(t) : 1 \leq i \leq d\}$,
3. For each $1 \leq i \leq d$, there exist $\ell_i > 1$, $\varepsilon > 0$ and a C^2 family φ_t of diffeomorphisms of $\mathbb{R}$ such that $\varphi_t(c_i(t)) = 0$, and $|f_t(x) - f_t(c_i(t))| = |\varphi_t(x) - \varphi_t(c_i(t))|^{\ell_i}$ holds when $|x - c_i(t)| < \varepsilon$ and $|t| < \delta$.

It is easy to see that if $f_t : [0,1] \to [0,1]$, $t \in [-1,1]$, is a C^3 family such that $f_0 \in \mathcal{A}$ has only non-degenerate critical points, then for $\delta > 0$ small enough, $(f_t)_{|t|<\delta}$ is a regular family. Besides, if f_t, $t \in [-1,1]$ with $f_0 \in \mathcal{A}$, is a real analytic family such that all the maps f_t have the same number of critical points, and the corresponding critical points have the same order, then f_t is regular.

1.2.2 Summability implies Collet-Eckmann almost surely

Several alternative proofs and generalizations of the Jakobson's theorem were obtained, see [4, 6–8, 12, 21, 22, 29, 30, 35–41]. In the first part of this thesis, we shall provide another generalization of the Jakobson's theorem. The following Theorem A comes from the recent paper [14] joint with Shen.

Theorem A. Consider a regular one-parameter family $f_t : [0,1] \to [0,1]$, $t \in [-1,1]$ and denote $F(x,t) = f_t(x)$. Assume

- $f_0 \in \mathcal{A}$ satisfies the summability condition (SC);
- the following non-degeneracy condition holds for $t = 0$.

$$(\mathrm{NV}_t) \qquad \sum_{j=0}^{\infty} \frac{\partial_t F(f_t^j(c), t)}{Df_t^j(f_t(c))} \neq 0 \text{ for every critical point } c \in \mathcal{C}(f_t).$$

Define

$$\mathscr{X} := \{t \in [-1,1] : f_t \text{ satisfies (CE), (SPR), (WR) and } (\mathrm{NV}_t)\}.$$

Then we have

$$\lim_{\varepsilon \to 0} \frac{|[-\varepsilon, \varepsilon] \cap \mathscr{X}|}{2\varepsilon} = 1.$$

In particular, $|\mathscr{X}| > 0$.

Remark. Note that if f_0 satisfies the condition (SC), then the summation in the condition (NV_t) converges at $t = 0$.

Like most of the approaches to the Jakobson's theorem, our proof is purely real analytic. Comparing to the previous works, our assumption on f_0 is much weaker and the result on strong polynomial recurrence condition is new. Previously the weakest assumption was given in [36], where f_0 satisfies (CE) and the critical points are at most sub-exponentially recurrent. Our analysis on the phase space geometry is based on the recent work [32], and these estimates are transformed to the parameter space by modifying the argument in [36].

For the family of real quadratic polynomials, our theorem is implicitly contained in [4], where complex method developed in [22] was applied to relate the phase and parameter spaces. The complex method is powerful for uni-critical maps, but does not work for multimodal maps.

The non-degeneracy condition (NV_t) was introduced in [36]. In [3], a geometric interpretation of this condition was given: for a real analytic family f_t of unimodal maps for which f_0 satisfies (SC), (NV_t) holds at $t = 0$ if and only f_t is transversal to the topological conjugacy class of f_0. In [18] and [2], it was proved that for the family of quadratic maps $Q_t(z) = z^2+t$, if Q_{t_0} satisfies (SC) then the condition (NV_t) automatically holds at $t = t_0$. By [23], for almost every $t \in \mathbb{R}$, Q_t is either uniformly hyperbolic or satisfies (SC). Thus our theorem gives a new proof of Theorem A and a part of Theorem B in [4].

Recently this transversality result has been generalized to higher degree polynomials in [19]. With this result, we can extend our Theorem A to the high dimensional version. More precisely, for any positive integer n and $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{C}^n$, let $P_{\mathbf{a}} = \sum\limits_{i=1}^{n} a_i x^i + \left(1 - \sum\limits_{i=1}^{n} a_i\right) x^{n+1}$. Hence, $P_{\mathbf{a}}(0) = 0$ and $P_{\mathbf{a}}(1) = 1$.

Let $\mathcal{P}$ be the collection of polynomial maps P with the following properties:

- $P([0,1]) \subset [0,1]$;
- $P|_{[0,1]} \in \mathcal{A}$ and $P|_{[0,1]}$ satisfies (SC).

Corollary 1.1. *Fix any positive integer n, we define parameter sets*

$$\Lambda = \{\boldsymbol{a} \in \mathbb{R}^n : P_{\boldsymbol{a}} \in \mathcal{P}\},$$

and

$$\Lambda_0 = \left\{ \boldsymbol{a} \in \Lambda : P_{\boldsymbol{a}}|_{[0,1]} \textit{ satisfies (CE) and (SPR) conditions } \right\}.$$

Then we have $\mathrm{Leb}_n(\Lambda \backslash \Lambda_0) = 0$.

In order to extend our result to the high dimensional version, we shall need the following lemma which is [3, Proposition 5.2]. We say that $p \in \mathbb{R}^n$ is a density point of a set X along a line l, if p is a density point of $l \cap X$ in l (endowed with the liner Lebesgue measure).

Lemma 1.2. *If $p \in \mathbb{R}^n$ is a density point of X along almost every line, then p is a density point of X in $\mathbb{R}^n$.*

Proof of Corollary 1.1. Consider the parameter set

$$\Lambda_1 = \{\mathbf{a} \in \mathbb{R}^n : P_{\mathbf{a}} \text{ has degenerate critical points}\}.$$

For any $\mathbf{a} \in \Lambda_1$, the discriminant $\Delta(\mathbf{a})$ of $P'_{\mathbf{a}}$ is equal to zero. Since $\Delta(\mathbf{a})$ is a polynomial in $\mathbf{a}$, the set Λ_1 has codimension one in $\mathbb{R}^n$, hence $\mathrm{Leb}_n(\Lambda_1) = 0$.

Define

$$\Pi = \{\mathbf{a} \in \mathbb{C}^n : \text{ all critical points of } P_{\mathbf{a}} \text{ are non-degenerate}\}.$$

Fix $\mathbf{a}_* \in \Pi$. For $\mathbf{a}$ in a small neighborhood of $\mathbf{a}_*$, the critical points of $P_{\mathbf{a}}$, which we denote by $c_1(\mathbf{a}), c_2(\mathbf{a}), \cdots, c_n(\mathbf{a})$, depend on $\mathbf{a}$ analytically.

Letting $v_j(\mathbf{a}) = P_{\mathbf{a}}(c_j(\mathbf{a}))$ for $j = 1, 2, \cdots, n$, $\{v_1(\mathbf{a}), v_2(\mathbf{a}), \cdots, v_n(\mathbf{a})\}$ is a local analytic coordinate, by [20, Proposition 1].

Now let $\mathbf{a}_* \in \Lambda \backslash \Lambda_1$. Suppose $c_1, c_2, \cdots, c_r$ are all the critical points of $P_{\mathbf{a}_*}$ in $(0, 1)$. By [19, Theorem 1], the rank of matrix

$$\mathbf{L} = (L(c_j, v_k))_{1\le j\le r, 1\le k\le n}$$

is equal to r, where

$$L(c_j, v_k) := \lim_{m\to\infty} \frac{\frac{\partial P^m_{\mathbf{a}}(c_j)}{\partial v_k}\Big|_{\mathbf{a}=\mathbf{a}_*}}{(P^{m-1}_{\mathbf{a}_*})'(P_{\mathbf{a}_*}(c_j))}.$$

Notice that $\{a_1, a_2, \cdots, a_n\}$ is a global analytic coordinate, then we define

$$L(c_j, a_k) := \lim_{m\to\infty} \frac{\frac{\partial P^m_{\mathbf{a}}(c_j)}{\partial a_k}\Big|_{\mathbf{a}=\mathbf{a}_*}}{(P^{m-1}_{\mathbf{a}_*})'(P_{\mathbf{a}_*}(c_j))}.$$

Hence, the rank of matrix

$$\widehat{\mathbf{L}} = (L(c_j, a_k))_{1\le j\le r, 1\le k\le n}$$

is equal to r and all entries of $\widehat{\mathbf{L}}$ are real numbers.

For any direction $\mathbf{u} \in S^{n-1}$, let $F^{(\mathbf{u})}(x, t) := P_{\mathbf{a}_*+t\mathbf{u}}(x)$, then we have

$$\sum_{m=0}^{\infty} \frac{\partial_t F^{(\mathbf{u})}(P^m_{\mathbf{a}_*}(c_j), 0)}{DP^m_{\mathbf{a}_*}(P_{\mathbf{a}_*}(c_j))} = (L(c_j, a_1), L(c_j, a_2), \cdots, L(c_j, a_k)) \cdot \mathbf{u}.$$

Thus, (NV_0) condition holds for $F^{(\mathbf{u})}(x,t)$ if and only if all entries of $\widehat{\mathbf{L}}\cdot\mathbf{u}$ are nonzero. Since the rank of matrix $\widehat{\mathbf{L}}$ is equal to r, all rows of matrix $\widehat{\mathbf{L}}$ are nonzero. If the k-th entry of $\widehat{\mathbf{L}}\cdot\mathbf{u}$ is equal to 0, then $\mathbf{u}$ is contained in the intersection of hyperplane in $\mathbb{R}^n$ and S^{n-1}. Thus, for almost all $\mathbf{u}$ in S^{n-1} (endowed with the Lebesgue measure on S^{n-1}), all entries of $\widehat{\mathbf{L}}\cdot\mathbf{u}$ are nonzero.

Hence, for almost every direction $\mathbf{u}$ in S^{n-1}, (NV_0) condition holds for one-parameter family $F^{(\mathbf{u})}(x,t)$. Together with our Theorem A, it follows that $\mathbf{a}_*$ is a density point of set Λ_0 along line $\mathbf{a}_*+t\mathbf{u}$. By Lemma 1.2, $\mathrm{Leb}_n((\Lambda\backslash\Lambda_1)\backslash\Lambda_0)=0$. Then the statement follows. $\square$

1.2.3 Asymptotic distributions of the critical orbits

As we know, an effective approach to study the dynamics of complicated systems is to describe the time average of the orbit by an acip. This naturally leads to investigating whether a point, especially the critical point, can be forecasted by an acip or not.

In the second part of this thesis, we shall study the asymptotic distributions of the critical orbits. Define a sequence of probability measures

$$\mu_n(x) := \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)},\ n=1,2,\cdots,$$

and, if this sequence converges to a probability measure μ as $n\to\infty$ in the sense of weak star topology, we say that μ is *the asymptotic distribution of the orbit of x for f*. In other words, we can say that x belongs to the basin of μ.

Theorem B. Consider a regular one-parameter family $f_t:[0,1]\to[0,1]$, $t\in[-1,1]$ and denote $F(x,t)=f_t(x)$. Define

$$\Delta=\{t\in[-1,1]:\ f_t\in\mathcal{A}\text{ and }f_t\text{ satisfies the conditions (SC) and (NV}_t)\}.$$

Let $\Delta_*\subset\Delta$ be the collection of parameters t with the following properties:

- f_t satisfies the conditions (CE) and (WR);
- for any $t\in\Delta_*$ and any $c\in\mathcal{C}(f_t)$, the asymptotic distribution of c for f_t exists and coincides with one of the ergodic acips.

Then we have $|\Delta \setminus \Delta_*| = 0$.

Several works in the direction of Theorem B were obtained. In [6], it was show that for the quadratic family $f_a(x) = 1 - ax^2$ on $(-1, 1)$ there is a set $\Delta_\infty \subset (1, 2)$ of a-values of positive Lebesgue measure for which f_a admits an acip and for which the critical point is typical with respect to this acip. In [10], it was proved that for almost every tent map, the critical points have the same distribution as the acip. Recently, this result were extended for the family of piecewise expanding unimodal maps, see [31]. Using technique from complex dynamics, it was shown that for a typical stochastic unimodal map the critical point is typical, see [5]. This technique allows to compare the phase space and the parameter space of a family of unimodal maps, but can not be applied in the multimodal case.

To prove Theorem B, we actually show that any Lebesgue density point t_0 of the set Δ is not a Lebesgue density point of the set $\Delta \setminus \Delta_*$. Let us summarize the main steps in the proof of Theorem B.

Step 1. Show that we can restrict Δ to the set of parameters t for which f_t admits a unique acip μ. A more precise version of this step is stated in Theorem B* in subsection 4.1.1. This reduction is based on the spectral decomposition which is Proposition 4.9.

Step 2. Fix $\phi \in C^2([0, 1], \mathbb{R})$ and $\delta > 0$. By a large deviation estimate, we obtain that

$$\left|\left\{x : \left|\frac{1}{N}\sum_{k=0}^{N-1}\phi(f_{t_0}^k(x)) - \int \phi d\mu_{t_0}\right| > \frac{\delta}{4}\right\}\right|$$

is exponentially small in N. As a consequence, we see that except for a set of exponentially small Lebesgue measure, $[0, 1]$ can be decomposed into a family of intervals $\{I_i\}_i$ such that $f^N : I_i \to [0, 1]$ is a diffeomorphism onto its image and such that

$$\left|\frac{1}{N}\sum_{k=0}^{N-1}\phi(f_{t_0}^k(x)) - \int \phi d\mu_{t_0}\right| < \frac{\delta}{4}$$

holds for each $x \in I_i$.

Step 3. Show that except for a set of polynomial small Lebesgue measure, I_i is "stable" under the small perturbations in the sense that for each t close to t_0, there exists an interval $I_i(t)$ close to I_i such that $f_t^N : I_i(t) \to [0, 1]$ is a diffeomorphism onto its image and

$$\left|\frac{1}{N}\sum_{k=0}^{N-1}\phi(f_t^k(x)) - \int \phi d\mu_{t_0}\right| < \frac{\delta}{2}$$

holds for each $x \in I_i(t)$.

Step 4. By a suitable choice of m and N, for each $c \in \mathcal{C}$, we show that if $f_t^{m+1}(c)$ belongs to a "stable" branch of time N, then t is a "good" parameter at time $m+N+1$.

1.3 Outline of the thesis

This thesis is organized as follows: In § 2, we provide some preliminaries that will be used in the subsequent discussions. We then prove Theorem A and Theorem B in § 3 and § 4 respectively.

Chapter 2

Preliminaries

2.1 Normalization

A regular family $g_t : [0,1] \to [0,1]$, $t \in [-1,1]$, is called *normalized* if the following hold:

(i) the maps g_t, $t \in [-1,1]$, all have the same critical points (denoted by $c_1, c_2, \ldots, c_d$);

(ii) there exists $\varepsilon_\# > 0$ and for each $i \in \{1, 2, \ldots, d\}$ there exists $\ell(c_i) > 1$ such that $|g_t(x) - g_t(c_i)| = |x - c_i|^{\ell(c_i)}$ holds for all $t \in [-1,1]$ and $x \in B(c_i, \varepsilon_\#)$;

(iii) $|\partial_t G(x,t)| \leq 1$ for all $x \in [0,1]$ and $|t| \leq 1$, where $G(x,t) = g_t(x)$.

To prove Theorem A, we only need to consider a normalized regular family. Indeed, given any regular family $f_t : [0,1] \to [0,1]$, $t \in [-1,1]$, one can find a C^2 family h_t, $t \in [-1,1]$, of diffeomorphisms from $[0,1]$ onto itself, such that $\widetilde{f}_t = h_t \circ f_t \circ h_t^{-1}$ all have the same critical points and are normalized as in (ii). Furthermore, take a small constant κ and define $g_t = \widetilde{f}_{t\kappa}$. Then the family $G(x,t) = g_t(x)$, $t \in [-1,1]$, satisfies all the properties (i), (ii), (iii). Note also that if f_0 satisfies (SC) then g_0 satisfies (SC); and if F satisfies the non-degeneracy condition (NV_t) at $t = 0$, then so does G.

To prove Theorem B, we also only need to consider a normalized regular family. Given any regular family $f_t; [0,1] \to [0,1]$, $t \in [-1,1]$, similarly, we can find $\widetilde{f}_t$ all have the same critical points and are normalized in (ii). For any $t_0 \in \Delta$, there exists $\kappa > 0$ such that $(g_t)_{t \in [-1,1]}$ is a normalized regular family, where $g_t = \widetilde{f}_{t_0 + t\kappa}$. This means that we can divide a regular family into the finite union of normalized regular families. If we can prove Theorem B for each normalized regular family, we can conclude that Theorem B holds for any regular family.

Unless otherwise stated, in this thesis, we assume that a one-parameter family $f_t : [0,1] \to [0,1]$, $t \in [-1,1]$ is a normalized regular family and denote $F(x,t) = f_t(x)$. Let $\mathcal{C}$ denote the common set of critical points of f_t, and let

$$\ell_{\max} = \max\{\ell(c) : c \in \mathcal{C}\}\ ,\ \ell_{min} = \min\{\ell(c) : c \in \mathcal{C}\}.$$

Moreover, let $f = f_0$ and $\mathrm{CV} = f(\mathcal{C})$. For each $c \in \mathcal{C}$ and $\delta > 0$, let

$$\widetilde{B}(c;\delta) = B(c, \delta^{1/\ell(c)}),\ D_c(\delta) = \frac{\delta}{|\widetilde{B}(c;\delta)|} = \frac{1}{2}\delta^{1-\ell(c)^{-1}},$$

and let

$$\widetilde{B}(\delta) = \bigcup_{c\in\mathcal{C}} \widetilde{B}(c;\delta).$$

Throughout, we fix $\delta_* = \delta_*(f) > 0$ such that the intervals $\widetilde{B}(c; 2\delta_*)$ are pairwise disjoint, and let

$$\mathrm{dist}_*(x,\mathcal{C}) = \begin{cases} \mathrm{dist}(f(x), \mathrm{CV}) & \text{if } x \in \widetilde{B}(\delta_*) \\ \delta_* & \text{otherwise.} \end{cases} \tag{2.1}$$

Replacing δ_* by a small constant, we may assume the following: for any $c \in \mathcal{C}$, $x \in \widetilde{B}(c;\delta_*)$ and $t \in [-1,1]$, we have

$$|f_t(x) - f_t(c)| = |x - c|^{\ell(c)}.$$

For any $x \in [0,1]$ and $n \in \mathbb{N}$, we define

$$\mathrm{A}(x,f,n) = \sum_{j=0}^{n-1} \frac{|Df^j(x)|}{\mathrm{dist}(f^j(x),\mathcal{C})}.$$

So if $f^j(x) \in \mathcal{C}$ for some $j < n$, then $\mathrm{A}(x,f,n) = \infty$. Note that for each $x \in [0,1]$, we have $\mathrm{dist}(x,\mathcal{C}) \le 1$.

2.2 Growth of derivatives along f_t-orbits

In this section, we study the derivative growth along f_t-orbits, the main result is the following Proposition 2.1.

Proposition 2.1. *Let $(f_t)_{t\in[-1,1]}$ be a normalized regular family of interval maps and $f \in \mathcal{A}$ satisfies the condition (SC). For each $\varepsilon > 0$ small enough, there exist $\Lambda(\varepsilon) > 1$ and $\alpha(\varepsilon) > 0$ such that*

$$\lim_{\varepsilon\to 0}\Lambda(\varepsilon) = \infty,\ \lim_{\varepsilon\to 0}\alpha(\varepsilon) = 0$$

and the following hold for $|t| < \varepsilon$:

(i) Let $x \in [0,1]$ be such that dist$(x, CV) \leq 4\varepsilon$, with $f_t^j(x) \notin \widetilde{B}(\varepsilon)$ for $j = 0, 1, \cdots, s-1$ and $f_t^s(x) \in \widetilde{B}(c; 2\varepsilon)$ for some $c \in \mathcal{C}$. Then

$$|Df_t^s(x)| \geq \frac{\Lambda(\varepsilon)}{D_c(\varepsilon)}\exp(\varepsilon^{\alpha(\varepsilon)}s) \tag{2.2}$$

(ii) Let $x \in [0,1]$ be such that $f_t^j(x) \notin \widetilde{B}(\varepsilon)$ for $j = 0, 1, \cdots, s-1$, then

$$|Df_t^s(x)| \geq A\varepsilon^{1-\ell_{\max}^{-1}}\exp(\varepsilon^{\alpha(\varepsilon)}s) \tag{2.3}$$

where $A > 0$ is a constant independent of ε.

(iii) Moreover, if f satisfies the condition (CE), we can replace $\varepsilon^{\alpha(\varepsilon)}$ by a positive constant α in the statements (i) and (ii).

Remark. The space $(f_t)_{t\in[-1,1]}$ is admissible in the sense of [32]. Thus by [32, Theorem 1], we have that the statements (i) and (ii) in Proposition 2.1 hold. The proof is based on decomposition of an f_t-orbit into pieces that can be shadowed by f_0-orbits and a delicate choice of the binding periods played an central role. We shall use the binding argument which has been introduced in [32] to prove the statement (iii). The statements (i) and (ii) will be used in § 3 and the statement (iii) will be used in § 4.

Definition 2.1. Let $(f_t)_{t\in[-1,1]}$ be a one-parameter C^1 family. Given $v \in [0,1]$ and $C > 0$, a positive integer N is called a C-binding period for (v, ε) if for each $y \in [0,1]$

with $|y - v| < \varepsilon$, each $|t| < \varepsilon$ and each $0 \le j < N$, the following hold:

$$2|f_t^j(y) - f^j(v)| < \operatorname{dist}(f^j(v), \mathcal{C}); \tag{2.4}$$

$$e^{-1}|Df^{j+1}(v)| \le |Df_t^{j+1}(y)| \le e|Df^{j+1}(v)|; \tag{2.5}$$

$$C\varepsilon|Df^{j+1}(v)| \ge |f_t^{j+1}(y) - f^{j+1}(v)|. \tag{2.6}$$

We shall need the following lemma which is [32, Lemma 2.3].

Lemma 2.2. *Let $(f_t)_{t\in[-1,1]}$ be a normalized regular family and $f \in \mathcal{A}$. Then there exists $\theta_1 > 0$ such that the following holds provided that $\varepsilon > 0$ is small enough. Let $v \in [0,1]$ and let N be a positive integer such that*

$$W = \sum_{j=0}^{N} |Df^j(v)|^{-1} < \infty \text{ and } A(v, f, N)W \le \theta_1/\varepsilon.$$

Then N is an eW-binding period for (v, ε).

We shall also need the following lemma which is [32, Proposition 4.1].

Lemma 2.3. *Given $f \in \mathcal{A}$ which satisfies the condition (SC), $L > 1$, $\theta \in (0,1)$ and $\zeta > 0$, for any critical value v and $\delta > 0$ small enough, there exists a positive integer $M_v(\delta)$ such that the following hold:*

$$A(v, f, M_v(\delta)) \le \theta/\delta, \tag{2.7}$$

$$f^j(v) \notin \widetilde{B}(L\delta) \text{ for each } j = 0, 1, \cdots, M_v(\delta) - 1, \tag{2.8}$$

and

$$|Df^{M_v(\delta)+1}(v)| \ge \left(\frac{\delta'}{\delta}\right)^{1-\zeta}, \tag{2.9}$$

where $\delta' = \max(d_*(f^{M_v(\delta)}(v), \mathcal{C}), \delta)$. Moreover, we have $M_v(\delta) \to \infty$ as $\delta \to 0$.

Given $f \in \mathcal{A}$ which satisfies the condition (CE), as in [32], we define a preferred binding period for each each $v \in \mathrm{CV}$ and each $\delta > 0$ small.

Let $C_0 = C_0(f) = \max_{[0,1]} |Df| \ge 1$, let $\eta_* = \eta_*(f)$ be a constant which is smaller than the distance between any two distinct critical points and let

$$W_0 = W_0(f) = \max_{v\in\mathrm{CV}} \sum_{n=0}^{\infty} |Df^n(v)|^{-1}.$$

Let $\theta > 0$ be a small constant such that

$$4\theta W_0 \le \theta_1 \text{ and } 16e\theta W_0 C_0 \le \eta_*, \tag{2.10}$$

where $\theta_1 > 0$ is as in Lemma 2.2. Moreover, fix constants $L > 2^{\ell_{max}+1}$ and $\zeta \in (0, \ell_{max}^{-1})$. For $v \in \mathrm{CV}$ and $\delta > 0$ small, we fix a positive integer $M_v(\delta)$, called *the preferred binding period for* (v, δ), such that the conclusion of Lemma 2.3 holds for these constants θ, L and ζ. Note that Lemmas 2.2 and 2.3 ensure that $M_v(\delta)$ is a eW_0-binding period for $(v, 4\delta)$. Furthermore, $M_v(\delta) \to \infty$ as $\delta \to 0$ for each $v \in \mathrm{CV}$. Combining this with the fact that f satisfies (CE), we have

$$\Lambda_0(\delta) := \inf_{v \in \mathrm{CV}} |Df^{M_v(\delta)+1}(v)| \to \infty \text{ as } \delta \to 0. \tag{2.11}$$

The following lemma is essentially proved in [32].

Lemma 2.4. *Consider a normalized regular family $(f_t)_{t \in [-1,1]}$ and $f \in \mathcal{A}$ satisfies the condition (CE). There exist positive constants α_0, ζ_1 and ζ_2 with the following property. For $\delta > 0$ sufficiently small and $v \in CV$, let $M = M_v(\delta) \ge 1$ be the preferred binding period defined as above. Then for any $|t| < \delta$ and $y \in [0,1]$ with $|y - v| < 4\delta$, we have*

$$y_j := f_t^j(y) \notin \widetilde{B}(2\delta) \text{ for all } 0 \le j < M; \tag{2.12}$$

$$|Df_t^M(y)| \ge \frac{\Lambda_0(\delta)^{\zeta_1}}{D_c(\delta)} e^{\alpha_0 M}, \tag{2.13}$$

where c is the critical point of f which is closest to y_M. Moreover, if $y_M \notin \widetilde{B}(\delta)$, then

$$|Df_t^{M+1}(y)| \ge \Lambda_0(\delta)^{\zeta_2} \left(\frac{dist_*(y_M, \mathcal{C})}{\delta} \right)^{1-\ell_{max}^{-1}} e^{\alpha_0(M+1)}. \tag{2.14}$$

Proof. Fix $v \in \mathrm{CV}$ and $\delta > 0$ small. Since M is a eW_0-binding period for $(v, 4\delta)$, by (2.4) and (2.8), the statement (2.12) holds provided that $\delta > 0$ is small enough. Furthermore, by (2.6), we have

$$|y_M - f^M(v)| \le 4e\delta W_0 |Df^M(v)|. \tag{2.15}$$

Since $\mathrm{dist}(f^{M-1}(v), \mathcal{C}) \le 1$ and $|Df(f^{M-1}(v))| \le C_0$, we obtain that

$$|Df^M(v)| \le |Df(f^{M-1}(v))| \cdot \frac{|Df^{M-1}(v)|}{\mathrm{dist}(f^{M-1}(v), \mathcal{C})} \le C_0 \mathrm{A}(v, f, M). \tag{2.16}$$

Combining (2.15), (2.16) and (2.7), we obtain

$$|y_M - f^M(v)| \le 4e\delta W_0 |Df^M(v)| \le 4e\delta W_0 C_0 \mathrm{A}(v, f, M) \le \eta_*/4, \tag{2.17}$$

so $|f^M(v) - c| \le 3\mathrm{dist}(f^M(v), \mathcal{C})$. Let $\delta' = \max(\mathrm{dist}_*(f^M(v), \mathcal{C}), \delta)$. We show that there exits a constant $C_1 > 1$ such that

$$|Df(f^M(v))| \le C_1 D_c(\delta'). \tag{2.18}$$

Indeed, if c is the critical point of f which is closest to $f^M(v)$, then (2.18) follows from the local behavior of f near c; otherwise, $\mathrm{dist}(f^M(v), \mathcal{C}) \ge \eta_*/4$, which implies that δ' is bounded from below by a constant. Thus, (2.18) holds. Replacing C_1 by a large constant, if necessary, we may assume that

$$|Df_t(y_M)| \ge C_1^{-1} D_c(\delta''), \tag{2.19}$$

where $\delta'' = \mathrm{dist}_*(y_M, \mathcal{C})$.

Let $\zeta_1 = (\ell_{max}^{-1} - \zeta)/(2 - 2\zeta)$. By (2.9), we obtain

$$|Df^{M+1}(v)| \ge |Df^{M+1}(v)|^{2\zeta_1} \left(\frac{\delta'}{\delta}\right)^{1-\ell_{max}^{-1}} \tag{2.20}$$

Let us prove the inequality (2.13). By (2.5), (2.18) and (2.20), we have

$$|Df_t^M(y)| \ge \frac{|Df^{M+1}(v)|}{e|Df(f^M(v))|} \ge \frac{|Df^{M+1}(v)|^{2\zeta_1}}{C_1 e D_c(\delta')} \left(\frac{\delta'}{\delta}\right)^{1-\ell_{max}^{-1}} \ge \frac{C_2 |Df^{M+1}(v)|^{2\zeta_1}}{D_c(\delta)},$$

where $C_2 > 0$ is a constant and we used $\delta' \ge \delta$ for the last inequality. Note that f satisfies (CE) and $\Lambda_0(\delta) \le |f^{M+1}(v)|$. Thus there exists a constant α_0 such that (2.13) holds provided that $\delta > 0$ is small enough.

Finally, let us assume $y_M \notin \widetilde{B}(\delta)$, which implies $\delta'' \ge \delta$, and prove that (2.14) holds with $\zeta_2 = \zeta_1/\ell_{max}$ provided that $\delta > 0$ is small enough. We distinguish two cases.

Case 1. $\delta'' > |Df^{M+1}(v)|^{2\zeta_1}\delta' \geq |Df^{M+1}(v)|^{2\zeta_1}\delta$. Note that $\delta'' \leq \delta_*$ and $|Df^{M+1}(v)| \to \infty$ as $\delta \to 0$. Thus, $\delta' < \delta_*$, which implies that $f^M(v) \in \widetilde{B}(\delta_*)$.

We show that there exists a constant C_3 such that

$$|y_M - f^M(v)| \geq C_3|\widetilde{B}(c;\delta'')|. \tag{2.21}$$

If c is the critical point of f which is closest to $f^M(v)$, by $\delta'' > |Df^{M+1}(v)|^{2\zeta_1}\delta'$, we have $|y_M - f^M(v)| > \frac{1}{2}|y_M - c|$, which implies that (2.21) holds; otherwise, $|y_M - f^M(v)|$ is bounded below by a constant. Combining with $\delta'' < \delta_*$, (2.21) holds.

Note that M is a eW_0-binding period for $(v, 4\delta)$, by (2.6), we have

$$4eW_0\delta|Df^M(v)| \geq |y_M - f^M(v)|.$$

Combining this with (2.5), (2.19) and (2.21), we obtain that

$$|Df_t^{M+1}(y)| \geq \frac{|Df^M(v)|}{e}|Df_t(y_M)| \geq \frac{|y_M - f^M(y)||Df_t(y_M)|}{4e^2W_0\delta} \geq C_4\frac{\delta''}{\delta},$$

where C_4 is a constant. Hence, we ge that

$$|Df_t^{M+1}(y)| \geq C_4|Df^{M+1}(v)|^{2\zeta_2}\left(\frac{\delta''}{\delta}\right)^{1-\ell_{max}^{-1}},$$

since $\delta'' \geq |Df^{M+1}(v)|^{2\zeta_1}\delta$. Note that f satisfies (CE). Thus, there exists a positive constant α_0 such that the inequality (2.14) holds provided that $\delta > 0$ is small enough.

Case 2. $\delta'' \leq |Df^{M+1}(v)|^{2\zeta_1}\delta'$. In this case, by (2.5), we obtain

$$|Df_t^{M+1}(y)| \geq \frac{|Df^{M+1}(v)|}{e}\frac{|Df_t(y_M)|}{|Df(f^M(v))|}.$$

Together with (2.18), (2.19) and (2.20), there exists a constant C_5 such that

$$\begin{aligned} |Df_t^{M+1}(y)| &\geq C_5|Df^{M+1}(v)|^{2\zeta_1}\left(\frac{\delta'}{\delta}\right)^{1-\ell_{max}^{-1}}\left(\frac{\delta''}{\delta'}\right)^{1-\ell(c)^{-1}} \\ &\geq C_5|Df^{M+1}(v)|^{2\zeta_1}\left(\frac{\delta'}{\delta''}\right)^{\ell(c)^{-1}-\ell_{max}^{-1}}\left(\frac{\delta''}{\delta}\right)^{1-\ell_{max}^{-1}} \\ &\geq C_5|Df^{M+1}(v)|^{2\zeta_1(1-\ell(c)^{-1}-\ell_{max}^{-1})}\left(\frac{\delta''}{\delta}\right)^{1-\ell_{max}^{-1}}, \end{aligned}$$

where for the last inequality we have used $\delta'' \leq |Df^{M+1}(v)|^{2\zeta_1}\delta'$. The inequality (2.14) follows provided that $\delta > 0$ is sufficiently small. □

For each $c \in \mathcal{C}$, each $\varepsilon > 0$ and each $\delta > 0$, let $\mathcal{L}_c^{\varepsilon}(\delta)$ denote the collection of all sequences $\{f_t^j(x)\}_{j=0}^n$ with $|t| < \varepsilon$, $f_t^j(x) \notin \widetilde{B}(\delta)$ for $j = 0, 1, \ldots, n-1$ and $f_t^n(x) \in \widetilde{B}(c;\delta)$. Then we have the following lemma which is [32, Lemma 5.3].

Lemma 2.5. *Consider a normalized regular family $(f_t)_{t\in[-1,1]}$ and $f \in \mathcal{A}$ satisfies the condition (SC). For each $\delta > 0$, there exist $\varepsilon = \varepsilon(\delta) > 0$ and $\eta = \eta(\delta) > 0$ such that for any sequence $\{f_t^j(x)\}_{j=0}^n \in \mathcal{L}_c^{\varepsilon}(\delta)$, $c \in \mathcal{C}$, we have*

$$|Df_t^n(x)| \geq \frac{\kappa}{D_c(\delta)}\left(\frac{\delta}{\delta''}\right)^{1-\ell_{max}^{-1}} e^{\eta n},$$

where $\delta'' = \max(dist(x, CV), \delta)$ and $\kappa > 0$ is a constant independent of δ.

Let $\mathcal{I}_c^{\varepsilon}(\delta, \widehat{\delta})$ denote the collection of all sequences $\{f_t^j(x)\}_{j=0}^n$ for which $|t| < \varepsilon$ and there exists $v \in \mathrm{CV}$ such that $|x - v| \leq 4\delta$ and such that one of the following holds:

- either $f_t^{M_v(\delta)}(x) \in \widetilde{B}(c;\widehat{\delta})$ and $n = M_v(\delta)$;
- or $f_t^{M_v(\delta)}(x) \notin \widetilde{B}(\widehat{\delta})$, $n > M_v(\delta)$ and $\{f_t^j(x)\}_{j=M_v(\delta)+1}^n \in \mathcal{L}_c^{\varepsilon}(\widehat{\delta})$.

We shall use the following lemma which is similar to [32, Lemma 5.4].

Lemma 2.6. *Consider a normalized regular family $(f_t)_{t\in[-1,1]}$ and $f \in \mathcal{A}$ satisfies the condition (CE). There exists $\zeta > 0$ such that the following holds. For each $\delta_0 > 0$ small enough, there exist $\varepsilon_0 > 0$ and $\eta_0 > 0$ such that for each $\{f_t^j(x)\}_{j=0}^n \in \mathcal{I}_c^{\varepsilon}(\delta, \delta_0)$*

with $\delta \in (0, \delta_0]$ *and* $0 \le \varepsilon \le \min(\varepsilon_0, \delta)$*, we have*

$$|Df_t^n(x)| \ge \frac{\Lambda_0(\delta)^\zeta}{D_c(\delta)} e^{\eta_0 n}. \tag{2.22}$$

Moreover, if $f_t^n(x) \notin \widetilde{B}(\varepsilon)$ *then*

$$|Df_t^{n+1}(x)| \ge \Lambda_0(\delta)^\zeta \left(\frac{dist_*(f_t^n(x), \mathcal{C})}{\delta}\right)^{1-\ell_{max}^{-1}} e^{\eta_0(n+1)}. \tag{2.23}$$

Proof. We assume $\delta_0 < \delta_*$ is a small constant such that the conclusion of Lemma 2.4 holds for any $\delta \in (0, \delta_0]$, and let ζ_1, ζ_2 and α_0 be the constants determined by Lemma 2.4. Let $\varepsilon_0 = \varepsilon(\delta_0)$ and $\eta = \eta(\delta_0)$ be the constants determined by Lemma 2.5, and let $\eta_0 = \min(\eta, \alpha_0)$ and $\zeta = \min(\zeta_1, \zeta_2)/2$.

For any $\{f_t^j(x)\}_{j=0}^n \in \mathcal{I}_c^\varepsilon(\delta, \delta_0)$ with $\delta \in (0, \delta_0]$ and $0 \le \varepsilon \le \min(\varepsilon_0, \delta)$, let $v \in \mathrm{CV}$ be such that $|x - v| \le 4\delta$, and let $M = M_v(\delta)$. We distinguish two cases.

Case 1. Assume $f_t^M(x) \in \widetilde{B}(\delta_0)$, then $n = M$. (2.22) holds by (2.13) in Lemma 2.4. If $f_t^n(x) \notin \widetilde{B}(\delta)$, then (2.23) holds by (2.14) in Lemma 2.4. Otherwise, we can obtain that $\mathrm{dist}_*(f_t^M(x), \mathcal{C}) < \delta$, which implies that

$$\frac{|Df_t(f_t^M(x))|}{D_c(\delta)} = \ell(c) \cdot \left(\frac{\mathrm{dist}_*(f_t^M(x), \mathcal{C})}{\delta}\right)^{1-\ell(c)^{-1}} \ge \ell(c) \cdot \left(\frac{\mathrm{dist}_*(f_t^M(x), \mathcal{C})}{\delta}\right)^{1-\ell_{max}^{-1}}.$$

Combining this with (2.13), (2.23) follows.

Case 2. Assume $f_t^M(x) \notin \widetilde{B}(\delta_0)$. Let $\delta' = \mathrm{dist}_*(f_t^M(x), \mathcal{C}) \ge \delta_0$ and let $\delta'' = \max(\mathrm{dist}(f_t^{M+1}(x), \mathrm{CV}), \delta_0)$. Then $\delta'' \le \delta' + \varepsilon \le 2\delta'$. By Lemma 2.5, we obtain

$$|Df_t^{n-M-1}(f_t^{M+1}(x))| \ge \frac{\kappa}{D_c(\delta_0)} \left(\frac{\delta_0}{\delta''}\right)^{1-\ell_{max}^{-1}} e^{\eta(n-M-1)}. \tag{2.24}$$

Together with (2.14) in Lemma 2.4, this implies

$$|Df_t^n(x)| \ge \frac{\kappa\Lambda_0(\delta)^{\zeta_2}}{2D_c(\delta_0)} \left(\frac{\delta_0}{\delta}\right)^{1-\ell_{max}^{-1}} e^{\eta_0 n} \tag{2.25}$$

Since $\delta_0 \ge \delta$ and $\delta > 0$ is small enough, the inequality (2.22) holds. Let $\rho = \mathrm{dist}_*(f_t^n(x), \mathcal{C}) \ge \varepsilon$. Since $\rho \le \delta_0 < \delta_*$, we have $|Df_t(f_t^n(x))| > D_c(\rho)$. Combining with (2.25) and $\rho < \delta_0$, the inequality (2.23) holds, provided that $\delta > 0$ is small enough. $\square$

We shall also need the following perturbation version of Máñe theorem, which is [32, Proposition 2.5].

Lemma 2.7. *Consider a normalized regular family $(f_t)_{t\in[-1,1]}$ and $f \in \mathcal{A}$. For any neighborhood U of $\mathcal{C}$, there exists $K > 1$ and $\widehat{\eta} > 0$ such that the following holds provided that $\varepsilon > 0$ is small enough. For any $x \in [0,1]$, $|t| < \varepsilon$, if $f_t^j(x) \notin U$ for all $0 \le j < n$, then $|Df_t^n(x)| \ge K^{-1}e^{\widehat{\eta}n}$.*

Proof of Proposition 2.1. The statements (i) and (ii) hold by [32, Theorem 1].

(iii) In the following, we assume that $(f_t)_{t\in[-1,1]}$ is a normalized regular family and f satisfies (CE). Let $\Lambda(\delta) = \Lambda_0(\delta)^\zeta$, where ζ is a constant determined by Lemma 2.6. It is easy to get that $\Lambda(\delta) \to \infty$ as $\delta \to \infty$. Let $\delta_0 < \delta_*$ be a small positive constant such that $\Lambda(\delta) > e$ for all $\delta \in (0, \delta_0]$. Replacing δ_0 by a small constant, we may assume that there exist positive constants $\varepsilon_0 < \delta_0$ and η_0 such that the conclusion of Lemma 2.6 holds. For each $c \in \mathcal{C}$ and $\varepsilon > 0$, let $\mathcal{R}_c(\delta)$ denote the collection of all sequences $\{f_t^j(x)\}_{j=0}^s$ with $|t| < \delta$, $\mathrm{dist}(x, \mathrm{CV}) < 4\delta$, $f_t^j(x) \notin \widetilde{B}(\delta)$ for $j = 0\ldots, s-1$ and $f_t^s(x) \in \widetilde{B}(c; 2\delta)$.

For each $0 \le \varepsilon \le \varepsilon_0$ and each $c \in \mathcal{C}$, consider $\{f_t^j(x)\}_{j=0}^s \in \mathcal{R}_c(\varepsilon)$ with $|x-v| \le 4\varepsilon$ for $v \in \mathrm{CV}$. We shall first prove that

$$|Df_t^s(x)| \ge \frac{\Lambda(\varepsilon)}{D_c(\varepsilon)} e^{\eta_0 s}. \tag{2.26}$$

Let s_1 be the minimal integer such that $s_1 \ge M_v(\varepsilon)$ and such that $f_t^{s_1}(x) \in \widetilde{B}(\delta_0)$. If $s_1 = s$ then the desired estimate follows from Lemma 2.6.

Assume $s_1 < s$. Then $\delta_1 = \mathrm{dist}_*(f_t^{s_1}(x), \mathcal{C}) \ge \varepsilon$. By Lemma 2.6, we have

$$|Df_t^{s_1+1}(x)| \ge \Lambda(\varepsilon) e^{\eta_0(s_1+1)} \left(\frac{\delta_1}{\varepsilon}\right)^{1-\ell_{max}^{-1}}.$$

Let $c_1 \in \mathcal{C}$ be such that $f_t^{s_1}(x) \in \widetilde{B}(c_1; \delta_0)$ and $v_1 = f(c_1)$. Define s_2 be the minimal integer such that $s_2 \ge M_{v_1}(\delta_1)$ and such that $f_t^{s_2}(x) \in \widetilde{B}(\delta_0)$. If $s_2 = s$ then we stop. Otherwise, we define c_2, v_2, δ_2 and s_3 similarly. The procedure continues until we get $s_k = s$. Then for each $i = 1, 2, \ldots, k-1$, $\{f_t^j(x)\}_{j=s_i+1}^{s_{i+1}} \in \mathcal{I}_{c_{i+1}}^\varepsilon(\delta_i, \delta_0)$. Thus by Lemma 2.6, we obtain

$$|Df_t^{s_{i+1}-s_i}(f_t^{s_i+1}(x))| \geq \Lambda(\delta_i)\left(\frac{\delta_{i+1}}{\delta_i}\right)^{1-\ell_{max}^{-1}} e^{\eta_0(s_{i+1}-s_i)},$$

for all $i = 1, 2, \ldots, k-2$ and

$$|Df_t^{s-s_{k-1}-1}(f^{s_{k-1}-1}(x))| \geq \frac{\Lambda(\delta_{k-1})}{D_c(\delta_{k-1})} e^{\eta_0(s-s_{k-1}-1)}.$$

From these inequalities and $\delta_{k-1} > \varepsilon$, the (2.26) holds.

Let $\widehat{\eta}$ be the constant given by Lemma 2.7 for $U = \widetilde{B}(\varepsilon_0)$, and let $\widehat{\varepsilon}$ be a small constant such that the conclusion of Lemma 2.7 holds for any $\varepsilon \leq \widehat{\varepsilon}$. Then let $\alpha = \min(\eta_0, \widehat{\eta}, 1)$. There are two parts in the statement (iii).

Part 1. For each $\varepsilon < \min(\varepsilon_0, \widehat{\varepsilon})$, each $c \in \mathcal{C}$ and $\{f_t^j(x)\}_{j=0}^s \in \mathcal{R}_c(\varepsilon)$, by (2.26), we have

$$|Df_t^s(x)| \geq \frac{\Lambda(\varepsilon)}{D_c(\varepsilon)} e^{\alpha s}.$$

Part 2. For each $\varepsilon < \min(\varepsilon_0, \widehat{\varepsilon})$ and each $\{f_t^j(x)\}_{j=0}^s$ with $f_t^j(x) \notin \widetilde{B}(\varepsilon)$ for $j = 0, 1, \ldots, s-1$, we need to show that

$$|Df_t^s(x)| \geq A\varepsilon^{1-\ell_{max}^{-1}} e^{\alpha s}, \tag{2.27}$$

where A is a constant independent of ε.

For each $j = 0, 1, \ldots, s$, let c_j be the critical point of f which is closest to $f_t^j(x)$ and let $\rho_j = \mathrm{dist}_*(f_t^j(x), \mathcal{C})$. By Lemma 2.7, (2.27) holds if $\rho_j \geq \varepsilon_0$ for all $j = 0, 1, \ldots, s-1$. Without loss of generality, we assume that $\rho_0 < \varepsilon_0$ and $\rho_{s-1} < \varepsilon_0$.

If there exists $s' < s-1$ such that $\rho_{s'} < \rho_{s-1}$, putting s' be the maximal integer with this property, we have $\{f_t^j(x)\}_{j=s'+1}^{s-1} \in \mathcal{R}_{c_{s-1}}(\rho_{s-1})$. By $\rho_{s-1} < \varepsilon_0$ and (2.26), we obtain

$$\begin{aligned} |Df_t^{s-s'-1}(f_t^{s'+1}(x))| &\geq \frac{\Lambda(\rho_{s-1})}{D_{c_{s-1}}(\rho_{s-1})} e^{\eta_0(s-s'-2)} |Df_t(f_t^{s-1}(x))| \\ &\geq e^{\alpha(s-s'-1)}. \end{aligned}$$

It follows that we only need to prove (2.27) under the further assumption that $\rho_{s-1} \leq \rho_j$ for all $0 \leq j < s$. In this case, let $s_0 < s_1 < \ldots < s_k = s-1$ be a sequence of integers such that $s_0 = 0$ and such that for each $0 \leq i < k$, s_{i+1} is the minimal integer such that $\rho_{s_{i+1}} \leq \rho_{s_i}$. Then $\{f_t^j(x)\}_{j=s_i+1}^{s_{i+1}} \in \mathcal{R}_{c_{s_{i+1}}}(\rho_{s_i})$, for $0 \leq i < k$. So by (2.26), we obtain

$$|Df_t^{s_{i+1}-s_i-1}(f_t^{s_i+1}(x))| \geq \frac{e^{\alpha(s_{i+1}-s_i)}}{D_{c_{s_{i+1}}}(\rho_{s_i})},$$

which implies that

$$\begin{aligned} |Df_t^s(x)| &\geq e^{\alpha(s-1)}|Df_t(x)| \prod_{i=1}^{k} \frac{D_{c_{s_i}}(\rho_{s_i})}{D_{c_{s_i}}(\rho_{s_{i-1}})} \\ &\geq A\rho_{s-1}^{1-\ell_{max}^{-1}} e^{\alpha s}, \end{aligned}$$

where $A > 0$ is a constant. Since $\rho_{s-1} \geq \varepsilon$, the inequality (2.27) holds. $\square$

Chapter 3

Summability implies Collet-Eckmann almost surely

The aim of this chapter is to prove Theorem A. In § 3.1, we state the Reduced Theorem A from which we deduce Theorem A. The rest of this chapter is devoted to the proof of the Reduced Theorem A. As described by Adrien Douady, the proof consists of two steps: in § 3.2 we "plough in the phase space" and in § 3.3 we "harvest in the parameter space".

3.1 Reduction

3.1.1 The (CE), (WR) and (PR) conditions

Define

$$q_\varepsilon(x) = \inf\left\{k \in \mathbb{N} : x \notin \bigcup_{c\in\mathcal{C}} \widetilde{B}(c; e^{-k\ell(c)}\varepsilon)\right\}. \tag{3.1}$$

Note that for $x \in \widetilde{B}(c;\varepsilon)$ with $\varepsilon > 0$ small and $c \in \mathcal{C}$, we have

$$\begin{aligned} |Df_t(x)| = \ell(c)\mathrm{dist}(x,\mathcal{C})^{\ell(c)-1} &\geq \ell(c)\left(e^{-q_\varepsilon(x)\ell(c)}\varepsilon\right)^{1-\ell(c)^{-1}} \\ &> e^{-q_\varepsilon(x)(\ell(c)-1)}D_c(\varepsilon) > e^{-q_\varepsilon(x)\ell_{\max}}D_c(\varepsilon). \end{aligned} \tag{3.2}$$

Thus the following is an immediate consequence of Proposition 2.1.

Lemma 3.1. *Provided that $\varepsilon > 0$ is small enough, the following holds: For any*

$y \in \widetilde{B}(\varepsilon)$, $t \in [-\varepsilon, \varepsilon]$, and $n \geq 1$, putting

$$m = \#\{1 \leq k \leq n : f_t^k(y) \in \widetilde{B}(\varepsilon)\},$$

we have

$$|Df_t^n(f_t(y))| \geq A\varepsilon^{1-\ell_{\max}^{-1}} \Lambda(\varepsilon)^m \exp\left(-\ell_{\max} \sum_{k=1}^{n} q_\varepsilon(f_t^k(y))\right) e^{\varepsilon^{\alpha(\varepsilon)} n}. \tag{3.3}$$

Furthermore, if $f_t^n(y) \in \widetilde{B}(\varepsilon)$, then

$$|Df_t^n(f_t(y))| \geq \Lambda(\varepsilon)^m \exp\left(-\ell_{\max} \sum_{k=1}^{n} q_\varepsilon(f_t^k(y))\right). \tag{3.4}$$

Proof. Let $0 = n_0 < n_1 < \cdots < n_m$ be all the integers in $\{0, 1, \ldots, n\}$ such that $f_t^{n_j}(y) \in \widetilde{B}(\varepsilon)$. Note that $\operatorname{dist}(f_t(y), \mathrm{CV}) \leq 2\varepsilon$. Applying Proposition 2.1 (i) to obtain lower bounds for $|Df_t^{n_{j+1}-n_j-1}(f_t^{n_j+1}(y))|$, $0 \leq j < m$, applying (ii) to obtain lower bounds for $|Df_t^{n-n_m-1}(f_t^{n_m+1}(y))|$ in the case $n_m < n$, and applying (3.2) give us the desired inequalities. $\square$

Remark. Lemma 3.1 which is based on Proposition 2.1, provides estimates of the derivatives along the critical orbits for maps near f_0. This is one of the key results in the phase space.

For $t \in [-1, 1]$, $\varepsilon > 0$ and $c \in \mathcal{C}$, let $S_1^{(c)}(t; \varepsilon) < S_2^{(c)}(t; \varepsilon) < \cdots < S_n^{(c)}(t; \varepsilon) < \cdots$ be the all positive integers such that $f_t^{S_j^{(c)}(t;\varepsilon)+1}(c) \in \widetilde{B}(\varepsilon)$, and let

$$d_j^{(c)}(t; \varepsilon) = q_\varepsilon(f_t^{S_j^{(c)}(t;\varepsilon)+1}(c)). \tag{3.5}$$

If c returns to $\widetilde{B}(\varepsilon)$ at most $n-1$ times, then let $S_n^{(c)}(t; \varepsilon) = \infty$ and $d_n^{(c)}(t; \varepsilon) = 0$.

3.1.2 Convention

Given $C > 0$, for each $n = 1, 2, \ldots$, we define

$$X_{n,\varepsilon}(C) = \left\{t \in [-\varepsilon, \varepsilon] : \sum_{j=1}^{k} d_j^{(c)}(t; \varepsilon) \leq Ck \text{ for any } k < n \text{ and } c \in \mathcal{C}\right\}, \tag{3.6}$$

and

$$X_\varepsilon(C) = \bigcap_{n=1}^{\infty} X_{n,\varepsilon}(C). \tag{3.7}$$

Given $C > 0$ and $\tau > 0$, for each $m = 0, 1, \ldots,$ we define

$$Y_\varepsilon^m(C,\tau) = \left\{ t \in X_\varepsilon(C) : \mathrm{dist}(f_t^{k+1}(\mathcal{C}), c) \geq \frac{\varepsilon^{1/\ell(c)}}{(k+1)^\tau} \text{ for } 0 \leq k < m \text{ and } c \in \mathcal{C} \right\} \tag{3.8}$$

and

$$Y_\varepsilon(C,\tau) = \bigcap_{m=0}^{\infty} Y_\varepsilon^m(C,\tau). \tag{3.9}$$

For any $\beta > 0$, we say that f satisfies *the condition (WR$_\beta$)*, if for each $c \in \mathcal{C}$, we have

$$\liminf_{\delta \to 0^+} \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(|Df(f^{i+1}(c))|) \cdot 1_{C(f,\delta)}(f^{i+1}(c)) \geq -\frac{1}{\beta}.$$

Note that if for each $\beta > 0$, f satisfies the condition (WR$_\beta$), then f satisfies the condition (WR).

Lemma 3.2. *(i) For any $C > 0$ and $\beta > 0$, if $t \in X_\varepsilon(C)$ and $\varepsilon > 0$ is small enough, then f_t satisfies the conditions (CE) and (WR$_\beta$).*

(ii) For any $C > 0$ and $\tau > 1$, if $t \in Y_\varepsilon(C,\tau)$ and $\varepsilon > 0$ is small enough, then f_t satisfies the condition (PR$_\tau$).

Proof. (i) For any $t \in X_\varepsilon(C)$, $n \geq 1$ and any $c \in \mathcal{C}$, let

$$m = \#\{1 \leq k \leq n : f_t^k(c) \in \widetilde{B}(\varepsilon)\}.$$

We claim that $m/n \geq 0$ is small, provided that $\varepsilon > 0$ is small enough. Indeed, we may assume that $m \geq 1$. Let $1 \leq n_1 < n_2 < \ldots < n_m \leq n$ be all the integers such that $f_t^{n_j}(c) \in \widetilde{B}(\varepsilon)$. By Lemma 3.1, we have

$$\begin{aligned} |Df_t^{n_m}(f_t(c))| &\geq \Lambda(\varepsilon)^m \exp\left(-\ell_{max} \sum_{j=1}^{m} d_j^{(c)}(t;\varepsilon)\right) \\ &\geq (\Lambda(\varepsilon) e^{-\ell_{max} C})^m. \end{aligned}$$

Note that $\Lambda(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Thus, we obtain that $m/n_m \geq 0$ is small, provided that $\varepsilon > 0$ is small enough. Since $n_m \le n$, the claim follows.

We first consider the condition (WR_β). For any $\varepsilon > 0$ small and $c' \in \mathcal{C}$, if $|x - c'| < \varepsilon^2$, we have

$$\log |Df_t(x)| > -(\ell(c') - 1)q_\varepsilon(x) + \log D_{c'}(\varepsilon) \ge -\ell_{max} q_\varepsilon(x).$$

Thus for any $\delta \in (0, \varepsilon^2)$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} \log(|Df_t(f_t^{i+1}(c))|) \cdot 1_{C(f_t,\delta)}(f_t^{i+1}(c)) \ge -\frac{\ell_{max}}{n}\sum_{j=1}^{m} d_j^{(c)}(t;\varepsilon) \ge -\ell_{max}\frac{Cm}{n}.$$

Combining this with the claim, f_t satisfies the condition (WR_β), provided that $\varepsilon > 0$ is small enough.

Let us consider the condition (CE). By lemma 3.1 again, we have

$$\begin{aligned} |Df_t^n(f_t(c))| &\ge A\varepsilon^{1-\ell_{max}^{-1}} e^{\varepsilon^{\alpha(\varepsilon)} n} \Lambda(\varepsilon)^m \exp\left(-\ell_{max}\sum_{j=1}^{m} d_j^{(c)}(t;\varepsilon)\right) \\ &\ge A\varepsilon^{1-\ell_{max}^{-1}} e^{\varepsilon^{\alpha(\varepsilon)} n}\left(\Lambda(\varepsilon)e^{-\ell_{\max}C}\right)^m \ge A\varepsilon^{1-\ell_{max}^{-1}} e^{\varepsilon^{\alpha(\varepsilon)} n}, \end{aligned}$$

provided that $\varepsilon > 0$ is small enough so that $\Lambda(\varepsilon) \ge e^{\ell_{\max}C}$. Hence, f_t satisfies the condition (CE). The first statement follows.

(ii) The second statement is trivial. □

Reduced Theorem A. Let $F = (f_t)$ be a normalized regular family of interval maps. Assume that $f_0 \in \mathcal{A}$ satisfies (SC) and that the condition (NV_t) holds at $t = 0$. Then

(i) Given $C > 0$ there exists $K = K(C) > 0$ such that $K(C) \to \infty$ as $C \to \infty$ and such that

$$|X_{n,\varepsilon}(C) \setminus X_{n+1,\varepsilon}(C)| \le K^{-n}\varepsilon, \; n = 1, 2, \ldots,$$

provided that $\varepsilon > 0$ is small enough.

(ii) Given $C > 0$, the following holds provided that $\varepsilon > 0$ is small enough: for any $t \in X_\varepsilon(C)$, (NV_t) holds.

(iii) Given $C > 0$, $\tau > \tau_0 > 1$, and $\sigma > 0$, we have

$$\left|Y_\varepsilon^m(C,\tau) \setminus Y_\varepsilon^{m+1}(C,\tau)\right| \leq \sigma\varepsilon(m+1)^{-\tau_0},\ m = 0, 1, , \ldots,$$

provided that $\varepsilon > 0$ is small enough.

Proof of Theorem A. For $\beta > 0$ and $\tau > 1$, let $\mathscr{Z}_{\beta,\tau}$ denote the set of parameters $t \in [-1, 1]$ for which f_t satisfies the conditions (CE), (NV$_t$), (WR$_\beta$) and (PR$_\tau$). We shall prove that

$$\lim_{\varepsilon \to 0} \frac{|\mathscr{Z}_{\beta,\tau} \cap [-\varepsilon, \varepsilon]|}{2\varepsilon} = 1. \tag{3.10}$$

Fix $\tau_0 \in (1, \tau)$ and $\eta > 0$. Choose a large constant $C > 0$ and a small constant $\sigma > 0$, such that

$$\frac{2K-3}{K-1} - \sigma \sum_{m=0}^{\infty} (m+1)^{-\tau_0} > 2 - \eta,$$

where $K = K(C)$ is as in the Reduced Theorem A. Provided that $\varepsilon > 0$ is small enough, we have

$$|X_\varepsilon(C)| = |X_{1,\varepsilon}(C)| - \sum_{n=1}^{\infty} |X_{n,\varepsilon}(C) \setminus X_{n+1,\varepsilon}(C)| \geq \frac{2K-3}{K-1} \cdot \varepsilon,$$

and

$$|Y_\varepsilon(C,\tau)| \geq |X_\varepsilon(C)| - \sum_{m=0}^{\infty} \left|Y_\varepsilon^m(C,\tau) \setminus Y_\varepsilon^{m+1}(C,\tau)\right| \geq (2-\eta)\,\varepsilon.$$

By Lemma 3.2, and the second statement of the Reduced Theorem A, we have $Y_\varepsilon(C,\tau) \subset \mathscr{Z}_{\beta,\tau}$, provided that $\varepsilon > 0$ is small enough. Thus

$$|\mathscr{Z}_{\beta,\tau} \cap [-\varepsilon, \varepsilon]| \geq (2-\eta)\varepsilon.$$

The equality (3.10) follows.

To complete the proof, we shall show that $\mathscr{Z}_{1,2} \setminus \mathscr{Z}$ has zero measure. Since $\mathscr{Z} = \bigcap_{k=1}^{\infty} \mathscr{Z}_{k,1+k^{-1}}$, we only need to show that for each $k > 1$, $\mathscr{Z}_{1,2} \setminus \mathscr{Z}_{k,1+k^{-1}}$ has measure zero. Indeed, for each $t_0 \in \mathscr{Z}_{1,2}$ and $k > 1$, we can apply the above argument to f_{t_0} instead of f_0, and obtain that t_0 is not a Lebesgue density point of $\mathscr{Z}_{1,2} \setminus \mathscr{Z}_{k,1+k^{-1}}$. By Lebesgue density Theorem, the statement follows. $\square$

3.1.3 Notations

We collect the notations which will be used in the rest of this chapter. For each $c \in \mathcal{C}$, let

$$W^{(c)} = \sum_{n=0}^{\infty} |Df^n(f(c))|^{-1} < \infty$$

and

$$a^{(c)} = \sum_{n=0}^{\infty} \frac{\partial_t F(f^n(c), 0)}{Df^n(f(c))} \neq 0.$$

As before let $S_j^{(c)}(t;\varepsilon)$ denote the j-th return time of $f_t(c)$ into $\widetilde{B}(\varepsilon)$, let $d_j^{(c)}(t;\varepsilon)$ be as defined in (3.5). Define

$$P_j^{(c)}(t;\varepsilon) = \frac{|Df_t^{S_j^{(c)}(t;\varepsilon)}(f_t(c))|}{\mathrm{dist}(f_t^{S_j^{(c)}(t;\varepsilon)+1}(c), \mathcal{C})},$$

$$p_j^{(c)}(t;\varepsilon) = \log \frac{P_j^{(c)}(t;\varepsilon)}{\mathrm{A}(f_t(c), f_t, S_j^{(c)}(t;\varepsilon))},$$

and

$$\widetilde{p}_j^{(c)}(t;\varepsilon) = \min\left\{p_j^{(c)}(t;\varepsilon), d_j^{(c)}(t;\varepsilon)\right\}.$$

3.2 Ploughing in the phase space

In this section, we obtain some estimates in the phase space. The main results are Propositions 3.3 and 3.8 below. Lemmas 3.6, 3.9 and 3.10 used in the argument are taken from [32]. Note that the non-degeneracy condition (NV_t) plays no role in this section.

3.2.1 A uniform summability

Proposition 3.3. *Given $\delta > 0$, the following holds provided that $\varepsilon > 0$ is small enough. For any $t \in [-\varepsilon, \varepsilon]$, $c \in \mathcal{C}$, $x \in [0,1]$ with $|x - f(c)| \leq 4\varepsilon$, if n is a non-negative integer such that $f_t^j(x) \notin \widetilde{B}(\varepsilon)$ holds for all $0 \leq j < n$, then*

$$\sum_{j=0}^{n} |Df_t^j(x)|^{-1} \leq W^{(c)} + \delta.$$

Before we prove this proposition, let us state a corollary.

Corollary 3.4. *Given $\theta > 0$ and $C > 0$ the following holds provided that $\varepsilon > 0$ is small enough: for each $t \in X_{n,\varepsilon}(C)$ and $c \in \mathcal{C}$, if S_n is the n-th return time of $f_t(c)$ into $\widetilde{B}(\varepsilon)$, then*

$$\sum_{i=0}^{S_n} |Df_t^i(f_t(c))|^{-1} \le W^{(c)} + \theta.$$

Proof. Denote $W = \max_{c \in \mathcal{C}} W^{(c)}$ and fix constants $\delta \in (0, \theta)$ and $\Lambda > (W + \theta)/(\theta - \delta)$. Let $S_0 = -1$, and for each $j \ge 1$, let S_j be the j-th return time of $f_t(c)$ into $\widetilde{B}(\varepsilon)$. Write $y_j = f_t^{S_j+1}(c)$, $x_j = f_t(y_j)$. Provided that $\varepsilon > 0$ is small enough, by Proposition 3.3, we have

$$H_k := \sum_{i=0}^{S_{k+1}-S_k-1} |Df_t^i(x_k)|^{-1} \le W + \delta,$$

for each $k = 1, 2, \ldots$. Moreover, by (3.4) in Lemma 3.1, we have

$$|Df_t^{S_k+1}(f_t(c))| \ge \Lambda(\varepsilon)^k e^{-\ell_{max} Ck} \ge \Lambda^k,$$

provided that $\varepsilon > 0$ is small enough. Thus

$$\begin{aligned}\sum_{i=0}^{S_n} |Df_t^i(f_t(c))|^{-1} &\le W^{(c)} + \delta + \sum_{k=1}^{n-1} |Df_t^{S_k+1}(f_t(c))|^{-1} H_k \\ &< W^{(c)} + \delta + \frac{W + \delta}{\Lambda - 1} < W^{(c)} + \theta,\end{aligned}$$

where for the last inequality we have used $\Lambda > (W + \theta)/(\theta - \delta)$. $\square$

Recall that δ_* is a small constant such that for any $c \in \mathcal{C}$, $x \in \widetilde{B}(c; \delta_*)$ and $t \in [-1, 1]$, we have

$$|f_t(x) - f_t(c)| = |x - c|^{\ell(c)}.$$

Fix $\varepsilon_0 \in (0, \delta_*/4]$ small such that Propositions 2.1 holds for all $\varepsilon \in (0, 4\varepsilon_0]$ with $\Lambda(\varepsilon) \ge 4$. For each $\varepsilon, \varepsilon' \in (0, 4\varepsilon_0]$ and $c \in \mathcal{C}$, let $\mathcal{D}^{(c)}(\varepsilon, \varepsilon')$ be the collection of all triples (x, t, n) with the following properties: $|x - f(c)| \le 4\varepsilon'$, $|t| \le \varepsilon$, and n is a non-negative integer such that $f_t^j(x) \notin \widetilde{B}(\varepsilon)$ for all $0 \le j < n$, and let

$$\widehat{L}^{(c)}(\varepsilon, \varepsilon') = \sup \left\{ \sum_{i=0}^{n} |Df_t^i(x)|^{-1} : (x, t, n) \in \mathcal{D}^{(c)}(\varepsilon, \varepsilon') \right\}.$$

Moreover, let

$$L^{(c)}(\varepsilon) = \widehat{L}^{(c)}(\varepsilon, \varepsilon),$$

$$L_*^{(c)}(\varepsilon) = \sup\{L^{(c)}(\varepsilon') : \varepsilon' \in [\varepsilon, 4\varepsilon_0]\}, \ L_*(\varepsilon) = \max_{c \in \mathcal{C}} L_*^{(c)}(\varepsilon),$$

and

$$\widehat{L}(\varepsilon, \varepsilon') = \max_{c \in \mathcal{C}} \widehat{L}^{(c)}(\varepsilon, \varepsilon').$$

Note that $L_*(\varepsilon)$ is decreasing in ε and $\widehat{L}(\varepsilon, \varepsilon')$ is increasing in ε'. Furthermore, by Proposition 2.1 (ii), $1 \le L_*(\varepsilon) < \infty$ for each $\varepsilon > 0$.

Lemma 3.5. *For any* $0 < \varepsilon \le \varepsilon' \le 2\varepsilon_0$, *we have*

$$\widehat{L}(\varepsilon, \varepsilon') \le 4L_*(\varepsilon) \left(\frac{\varepsilon'}{\varepsilon}\right)^{1-\ell_{max}^{-1}} \tag{3.11}$$

Proof. It suffices to prove that for any integer $k \ge 0$ such that $2^k\varepsilon \le 4\varepsilon_0$, we have

$$\widehat{L}(\varepsilon, 2^k\varepsilon) \le 2L_*(\varepsilon) \cdot 2^{k(1-\ell_{max}^{-1})}. \tag{3.12}$$

Indeed, this implies that for any $\varepsilon' \in [2^{k-1}\varepsilon, 2^k\varepsilon]$, we have

$$\widehat{L}(\varepsilon, \varepsilon') \le \widehat{L}(\varepsilon, 2^k\varepsilon) \le 2L_*(\varepsilon) \cdot 2^{k(1-\ell_{max}^{-1})} < 4L_*(\varepsilon) \left(\frac{\varepsilon'}{\varepsilon}\right)^{1-\ell_{max}^{-1}}. \tag{3.13}$$

Let us prove (3.12) by induction on k. By definition, the case $k = 0$ is clear. Assume (3.12) holds for all k not greater than some j. Let us consider the case $k = j+1$ with $2^{j+1}\varepsilon \le 4\varepsilon_0$. For $c \in \mathcal{C}$ and $(x, t, n) \in \mathcal{D}^{(c)}(\varepsilon, 2^{j+1}\varepsilon)$, we need to prove that

$$\sum_{i=0}^{n} |Df_t^i(x)|^{-1} \le 2L_*(\varepsilon) \cdot 2^{(j+1)(1-\ell_{max}^{-1})}. \tag{3.14}$$

If $f_t^i(x) \notin \widetilde{B}(2^{j+1}\varepsilon)$ holds for all $0 \le i < n$, then $(x, t, n) \in \mathcal{D}^{(c)}(2^{j+1}\varepsilon, 2^{j+1}\varepsilon)$, so (3.14) holds by definition of L_*. Otherwise, let $m \in \{1, 2, \ldots, n-1\}$ be minimal such that $f_t^m(x) \in \widetilde{B}(2^{j+1}\varepsilon)$. Then

$$\sum_{i=0}^{m} |Df_t^i(x)|^{-1} \leq L_*(2^{j+1}\varepsilon) \leq L_*(\varepsilon).$$

Let $c_* \in \mathcal{C}$ be the critical point closest to $f_t^m(x)$ and $\varepsilon_* = |f_t^{m+1}(x) - f_t(c_*)|$. Since $f_t^m(x) \notin \widetilde{B}(\varepsilon)$, it follows that $\varepsilon \leq \varepsilon_*$. By Proposition 2.1 (i), we have

$$|Df_t^{m+1}(x)| \geq \Lambda(2^{j+1}\varepsilon) \left(\frac{\varepsilon_*}{2^{j+1}\varepsilon}\right)^{1-\ell(c_*)^{-1}} \geq 4 \left(\frac{\varepsilon_*}{2^{j+1}\varepsilon}\right)^{1-\ell_{max}^{-1}}.$$

Note that $|f_t^{m+1}(x) - f(c_*)| \leq 2\varepsilon_*$. If $\varepsilon_*/2 \leq \varepsilon$, then $|f_t^{m+1}(x) - f(c_*)| \leq 4\varepsilon$, which implies that $(f_t^{m+1}(x), t, n-m-1) \in \mathcal{D}^{(c_*)}(\varepsilon, \varepsilon)$. Thus, we have that

$$\sum_{i=0}^{n-m-1} |Df_t^i(f_t^{m+1}(x))|^{-1} \leq L_*(\varepsilon) \leq 4L_*(\varepsilon) \left(\frac{\varepsilon_*}{2\varepsilon}\right)^{1-\ell_{max}^{-1}},$$

where for the last inequality we have used $\varepsilon_* \geq \varepsilon$. Otherwise, $\varepsilon < \varepsilon_*/2 \leq 2^j\varepsilon$, by induction and (3.13), we have

$$\sum_{i=0}^{n-m-1} |Df_t^i(f_t^{m+1}(x))|^{-1} \leq \widehat{L}(\varepsilon, \varepsilon_*/2) \leq 4L_*(\varepsilon) \left(\frac{\varepsilon_*}{2\varepsilon}\right)^{1-\ell_{max}^{-1}}.$$

Thus,

$$\begin{aligned}\sum_{i=0}^{n} |Df_t^i(x)|^{-1} &= \sum_{i=0}^{m} |Df_t^i(x)|^{-1} + |Df_t^{m+1}(x)|^{-1} \sum_{i=0}^{n-m-1} |Df_t^i(f_t^{m+1}(x))|^{-1} \\ &< 2L_*(\varepsilon) 2^{(j+1)(1-\ell_{max}^{-1})}.\end{aligned}$$

□

To complete the proof, we shall need the following result which is a reformulation of [32, Proposition 5.2].

Lemma 3.6. *For $\varepsilon > 0$ sufficiently small and each $c \in \mathcal{C}$, there exist a constant $\Lambda_0(\varepsilon) > 0$ and a positive integer $M = M_c(\varepsilon) \geq 1$ such that $\lim_{\varepsilon \to 0} \Lambda_0(\varepsilon) = \infty$ and such that the following holds: for any $t \in [-\varepsilon, \varepsilon]$ and $y \in [0,1]$ with $|y - f(c)| \leq 4\varepsilon$,* *we have*

$$y_j := f_t^j(y) \notin \widetilde{B}(2\varepsilon) \text{ for all } 0 \le j < M; \tag{3.15}$$

$$e^{-1}|Df^j(f(c))| \le |Df_t^j(y)| \le e|Df^j(f(c))| \text{ for all } 0 \le j \le M. \tag{3.16}$$

If $f_t^M(y) \notin \widetilde{B}(\varepsilon_0)$, *then*

$$|Df_t^{M+1}(y)| \ge \Lambda_0(\varepsilon)\left(\frac{\varepsilon_0}{\varepsilon}\right)^{1-\ell_{max}^{-1}}; \tag{3.17}$$

If $f_t^M(y) \in \widetilde{B}(\varepsilon_0)$ *and* $f_t^M(y) \notin \widetilde{B}(\varepsilon)$, *then*

$$|Df_t^{M+1}(y)| \ge \Lambda_0(\varepsilon)\left(\frac{dist(f_t^{M+1}(y), CV)}{\varepsilon}\right)^{1-\ell_{max}^{-1}}. \tag{3.18}$$

Lemma 3.7. *Let* $\delta > 0$ *be given. Then for* $\varepsilon > 0$ *small enough, and any* $c \in \mathcal{C}$,

$$L^{(c)}(\varepsilon) \le W^{(c)} + \delta L_*(\varepsilon).$$

Proof. In the following, we assume $\varepsilon > 0$ small. We need to prove that for each $(x, t, n) \in \mathcal{D}^{(c)}(\varepsilon, \varepsilon)$,

$$\sum_{j=0}^{n} |Df_t^j(x)|^{-1} \le W^{(c)} + \delta L_*(\varepsilon). \tag{3.19}$$

Let $M = M_c(\varepsilon)$ be as in Lemma 3.6. We first prove

$$\sum_{j=0}^{\min(n,M)} |Df_t^j(x)|^{-1} \le W^{(c)} + \delta L_*(\varepsilon)/2. \tag{3.20}$$

Take N large enough such that

$$\sum_{j=0}^{N} |Df^j(f(c))|^{-1} \ge W^{(c)} - \delta/(4e).$$

By continuity, we have

$$\sum_{j=0}^{\min(n,N)} |Df_t^j(x)|^{-1} \le W^{(c)} + \delta/4. \tag{3.21}$$

So (3.20) holds when $\min(n, M) \le N$. If $\min(n, M) > N$, then by (3.16),

$$\sum_{j=N+1}^{\min(n,M)} |Df_t^j(x)|^{-1} \le e \sum_{j=N+1}^{M} |Df^j(f(c))|^{-1} \le \delta/4 \le \delta L_*(\varepsilon)/4, \tag{3.22}$$

since $L_*(\varepsilon) \ge 1$. Together with (3.21), this implies (3.20).

In particular, (3.19) holds if $n \le M$. Let us assume now that $n > M$, so that $f_t^M(x) \notin \widetilde{B}(\varepsilon)$. To complete the proof, we need to prove that

$$\sum_{j=M+1}^{n} |Df_t^j(x)|^{-1} \le \delta L_*(\varepsilon)/2. \tag{3.23}$$

We distinguish two cases.

Case 1. Assume $f_t^M(x) \in \widetilde{B}(c_*; \varepsilon_0)$ for some $c_* \in \mathcal{C}$. Since $(x, t, n) \in \mathcal{D}^{(c)}(\varepsilon, \varepsilon)$, we have that $f_t^j(x) \notin \widetilde{B}(\varepsilon)$ hold for all $0 \le j < n$. Let $\varepsilon_* := |f_t^{M+1}(x) - f(c_*)|$. Then $\varepsilon_* \in [\varepsilon, 2\varepsilon_0]$ and $(f_t^{M+1}(x), t, n - M - 1) \in \mathcal{D}^{(c_*)}(\varepsilon, \varepsilon_*)$. By Lemma 3.5, we have

$$\sum_{j=M+1}^{n} |Df_t^j(x)|^{-1} \le \frac{\widehat{L}(\varepsilon, \varepsilon_*)}{|Df_t^{M+1}(x)|} \le \frac{4L_*(\varepsilon)}{|Df_t^{M+1}(x)|} \left(\frac{\varepsilon_*}{\varepsilon}\right)^{1-\ell_{max}^{-1}}.$$

Together with (3.18), this implies that

$$\sum_{j=M+1}^{n} |Df_t^j(x)|^{-1} \le 4\Lambda_0(\varepsilon)^{-1} L_*(\varepsilon) < \delta L_*(\varepsilon)/2.$$

Case 2. Assume $f_t^M(x) \notin \widetilde{B}(\varepsilon_0)$. Let k be the maximal integer with $M < k \le n$ and such that $f_t^j(x) \notin \widetilde{B}(\varepsilon_0)$ for all $M < j < k$. By Proposition 2.1 (ii), there exists a constant $C > 0$ such that

$$\sum_{j=0}^{k-M-1} |Df_t^j(f_t^{M+1}(x))|^{-1} \le C \tag{3.24}$$

$$|Df_t^{k-M-1}(f_t^{M+1}(x))| \ge 1/C. \tag{3.25}$$

Thus by (3.17) and (3.24),

$$\sum_{j=M+1}^{k} |Df_t^j(x)|^{-1} \le C|Df_t^{M+1}(x)|^{-1} \le \frac{C}{\Lambda_0(\varepsilon)} \left(\frac{\varepsilon}{\varepsilon_0}\right)^{1-\ell_{\max}^{-1}} < \delta L_*(\varepsilon)/4. \tag{3.26}$$

In particular, (3.23) holds if $k = n$. Assume that $k < n$. Then there exists $c_* \in \mathcal{C}$ such that $f_t^k(x) \in \widetilde{B}(c_*; \varepsilon_0)$. Since $(x, t, n) \in \mathcal{D}^{(c)}(\varepsilon, \varepsilon)$, we have that $f_t^j(x) \notin \widetilde{B}(\varepsilon)$ hold for all $0 \le j < n$. Let $\varepsilon_* := |f_t^{k+1}(x) - f_t(c_*)| \in [\varepsilon, \varepsilon_0]$. Then

$$(f_t^{k+1}(x), t, n-k-1) \in \mathcal{D}^{(c_*)}(\varepsilon, \varepsilon_*).$$

So by Lemma 3.5

$$\sum_{j=k+1}^{n} |Df_t^j(x)|^{-1} \le |Df_t^{k+1}(x)|^{-1} 4L_*(\varepsilon) \left(\frac{\varepsilon_*}{\varepsilon}\right)^{1-\ell_{\max}^{-1}}.$$

On the other hand,

$$|Df_t(f_t^k(x))| = \ell_{c_*} \varepsilon_*^{1-\ell_{c_*}^{-1}} \ge \varepsilon_*^{1-\ell_{\max}^{-1}},$$

so by (3.25) and (3.17),

$$|Df_t^{k+1}(x)| = |Df_t^{M+1}(x)||Df_t^{k-M-1}(f_t^{M+1}(x))||Df_t(f_t^k(x))| \ge \frac{\Lambda_0(\varepsilon)\varepsilon_0^{1-\ell_{\max}^{-1}}}{C} \left(\frac{\varepsilon_*}{\varepsilon}\right)^{1-\ell_{\max}^{-1}}.$$

Therefore,

$$\sum_{j=k+1}^{n} |Df_t^j(x)|^{-1} \le \frac{4CL_*(\varepsilon)}{\Lambda_0(\varepsilon)\varepsilon_0^{1-\ell_{\max}^{-1}}} \le \delta L_*(\varepsilon)/4.$$

Together with (3.26), this implies (3.23). This completes the proof of the lemma. □

Proof of Proposition 3.3. By Lemma 3.7, it suffices to show that $L_*(\varepsilon)$ is uniformly bounded. Arguing by contradiction, assume that this is not the case. As $L_*^{(c)}(\varepsilon)$ is monotone decreasing in ε for each c, it follows that $L_*(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Let us define a sequence of positive real numbers $\{\varepsilon_k\}_{k=1}^{\infty}$ inductively as follows. Note that $\varepsilon_0 > 0$ is a small constant. For each $k \in \mathbb{N}$, let $\varepsilon_{k+1} \le \varepsilon_k/2$ be the maximal real number with the following property: there exists $c_{k+1} \in \mathcal{C}$ such that $L^{(c_{k+1})}(\varepsilon_{k+1}) \ge L_*(\varepsilon_k/2)$. By our construction, we obtain that $\varepsilon_k \to 0$ as $k \to \infty$ and $2L^{(c_k)}(\varepsilon_k) \ge L_*(\varepsilon_k)$ holds for each k. Replacing $\{\varepsilon_k\}_{k=1}^{\infty}$ by its subsequence if necessary, we may assume that there exists $c \in \mathcal{C}$ such that $2L^{(c)}(\varepsilon_k) \ge L_*(\varepsilon_k)$. However, by Lemma 3.7, we have

$$L_*(\varepsilon_k) \le 2L^{(c)}(\varepsilon_k) \le 2W^{(c)} + \frac{1}{2}L_*(\varepsilon_k),$$

provided that k is large enough. It follows that $L_*(\varepsilon_k) \le 4W^{(c)}$, a contradiction. □

3.2.2 Essential returns

In this subsection, we will introduce three types of returns: essential, inessential and bound. The essential returns will play a prominent role in the proof of Reduced Theorem A. As we will show in Lemma 3.13, an important fact about the essential return is that the total depth of the inessential and bound returns is smaller than the depth of the essential return preceding them.

Definition 3.1. We say that $S_n^{(c)}(t;\varepsilon)$ is an *essential return time* of $f_t(c)$ into $\widetilde{B}(\varepsilon)$ if

$$P_n^{(c)}(t;\varepsilon) \geq 3^{n-k} P_k^{(c)}(t;\varepsilon), \text{ for all } 1 \leq k < n.$$

Given $C_0 > 0$, we define

$$\mathcal{T}_{\text{ess}}^{(c)}(t;\varepsilon) = \{k \geq 1 : S_k^{(c)}(t;\varepsilon) \text{ is an essential return time of } f_t(c) \text{ into } \widetilde{B}(\varepsilon)\},$$

and

$$\widehat{\mathcal{T}}_{\text{ess}}^{(c)}(C_0,t;\varepsilon) = \{k \in \mathcal{T}_{\text{ess}}^{(c)}(t;\varepsilon) : \widetilde{p}_k^{(c)}(t;\varepsilon) > C_0\}.$$

Refer to the end of section 2 for the definition of the notations $P_n^{(c)}$, $\widetilde{p}_k^{(c)}$, etc.

The goal of this section is to prove the following:

Proposition 3.8. *Given $C > 0, C_0 > 0, \tau > 1$ and $\gamma \in (0,1)$, the following hold provided that $\varepsilon > 0$ is small enough:*

(i) For $t \in X_{n,\varepsilon}(C) \setminus X_{n+1,\varepsilon}(C)$, $n = 1,2,\ldots$, there exists $c \in \mathcal{C}$ such that

$$\sum_{k \in \widehat{\mathcal{T}}_{ess}^{(c)}(C_0,t;\varepsilon), k \leq n} \widetilde{p}_k^{(c)}(t;\varepsilon) \geq (\gamma C - C_0)n.$$

(ii) For $t \in Y_\varepsilon^m(C,\tau) \setminus Y_\varepsilon^{m+1}(C,\tau)$, $m = 0,1,2,\ldots$, there exists $c \in \mathcal{C}$ and $n \in \widehat{\mathcal{T}}_{ess}^{(c)}(C_0,t;\varepsilon)$ such that $m = S_n^{(c)}(t;\varepsilon)$ and

$$p_n^{(c)}(t;\varepsilon) \geq \gamma\tau \log(m+1).$$

We shall need the following lemma which is [32, Proposition 5.6].

Lemma 3.9. *For any $\varepsilon > 0$ small enough, there exists a constant $\kappa(\varepsilon) > 0$ such that for $|t| \leq \varepsilon$ and $x \in [0,1]$, if n is an integer such that $f_t^j(x) \notin \widetilde{B}(\varepsilon)$ for $0 \leq j < n$*

and $f_t^n(x) \in \widetilde{B}(c;\varepsilon)$ *for some* $c \in \mathcal{C}$*, then*

$$A(x, f_t, n) \le \kappa(\varepsilon) \cdot \frac{|Df_t^n(x)|}{|\widetilde{B}(c;\varepsilon)|} \tag{3.27}$$

and such that

$$\kappa(\varepsilon) \to 0 \text{ as } \varepsilon \to 0 \tag{3.28}$$

We shall also need the following lemma which is [32, Lemma 2.1].

Lemma 3.10. *There exists a constant* $\theta_0 > 0$ *such that for any* $(x,t) \in [0,1]\times[-1,1]$ *and any integer* $n \ge 1$ *with* $A(x, f_t, n) < \infty$*, putting*

$$J = \left[x - \frac{\theta_0}{A(x, f_t, n)}, x + \frac{\theta_0}{A(x, f_t, n)}\right] \cap [0,1],$$

we have that $f_t^n|J$ *is a diffeomorphism and*

$$\exp\left(-\frac{|x-y|}{|J|}\right) \le \frac{|Df_t^j(x)|}{|Df_t^j(y)|} \le \exp\left(\frac{|x-y|}{|J|}\right)$$

holds for all $y \in J$ *and* $0 \le j \le n$.

In the following, fix $C > 1$, $\gamma \in (0,1)$ and denote $\rho = 1 - \sqrt{\gamma}$, $\rho_1 = \rho/4$, $\rho_2 = \rho_1/(2\ell_{max})$. Let $\varepsilon > 0$ denote a small constant and we fix a parameter $t \in [-\varepsilon, \varepsilon]$. For simplicity, we shall drop t, ε from the notations. So $S_i^{(c)} = S_i^{(c)}(t;\varepsilon)$, $d_i^{(c)} = d_i^{(c)}(t;\varepsilon)$, etc.

Free returns

For each $c \in \mathcal{C}$, we investigate the orbit of $f_t(c)$. If $f_t^{S_i^{(c)}+1}(c)$ falls in a tight vicinity of some critical point c', we expect the the orbit of $f_t^{S_i^{(c)}+2}(c)$ can be shadowed by the orbit of $f_t(c')$ at least for some period of time.

With this purpose, we define

$$\widehat{S}_i^{(c)} = \sup\{S > S_i^{(c)} : \mathrm{A}(f_t^{S_i^{(c)}+2}(c), f_t, S - S_i^{(c)}) \le \theta_0 e^{(d_i^{(c)}-1)\ell(c')}\varepsilon^{-1}\},$$

and

$$\widetilde{S}_i^{(c)} = \inf\{S > \widehat{S}_i^{(c)} : f_t^{S+1}(c) \in \widetilde{B}(\varepsilon)\},$$

where c' denote the critical point of f which is closest to $f_t^{S_i^{(c)}+1}(c)$.

We define positive integers $i_1 < i_2 < \cdots$ in the following way: $i_1 = 1$. Once i_k and $S_{i_k}^{(c)}$ are both well-defined, let i_{k+1} be such that $\widetilde{S}_{i_k}^{(c)} = S_{i_{k+1}}^{(c)}$. The procedure stops whenever $S_{i_k}^{(c)}$ or $\widetilde{S}_{i_k}^{(c)}$ is not well-defined. The positive integers $S_{i_k}, k = 1, 2, \ldots$ are called *free return times* of $f_t(c)$ into $\widetilde{B}(\varepsilon)$.

Note that nothing prevents the orbit of $f_t(c)$ from entering into $\widetilde{B}(\varepsilon)$ between two consecutive free returns. These returns are called the *bound returns*. The following lemma will show that the total depth of the bound returns is smaller than the depth of the free return preceding them and that the loss in the growth of the derivative caused by the free return have been compensated before the next free return.

Lemma 3.11. *Consider $t \in X_{n,\varepsilon}(C)$, $c \in \mathcal{C}$ and $1 \leq i < n$. Then*

$$\sum_{k=S_i^{(c)}+2}^{\widehat{S}_i^{(c)}+1} q_\varepsilon(f_t^k(c)) < \rho_1 \cdot d_i^{(c)}. \tag{3.29}$$

Moreover, if there exists $j \leq n$ such that $S_j^{(c)} = \widetilde{S}_i^{(c)}$, then

$$\log \frac{P_j^{(c)}}{P_i^{(c)}} > d_j^{(c)} - \rho_1 d_i^{(c)} + (\log 3) \cdot (j - i). \tag{3.30}$$

Proof. Assume $\varepsilon > 0$ small and let $a = 2\ell_{max}/(\ell_{min} - 1)$, $\varepsilon' = e^a \varepsilon$. Let c_k denote the critical point of f which is closest to $f_t^{S_k^{(c)}+1}(c)$. For simplicity of notation, we shall write $S_k = S_k^{(c)}$, $\widehat{S}_k = \widehat{S}_k^{(c)}$ and $d_k = d_k^{(c)}$ for each k. Let $y = f_t^{S_i+1}(c)$, $x = f_t(y)$, $v = f_t(c)$ and $v_i = f_t(c_i)$. Note that $\mathrm{A}(x, f_t, \widehat{S}_i - S_i) \leq \theta_0/|v_i - x|$. So by Lemma 3.10, for $0 \leq k < \widehat{S}_i - S_i$, we have

$$e^{-1}|Df_t^{k+1}(x)| \leq |Df_t^{k+1}(v_i)| \leq e|Df_t^{k+1}(x)|, \tag{3.31}$$

and

$$|Df_t^{k+1}(x)| \geq e^{-1} \frac{\mathrm{dist}(f_t^{k+1}(v_i), f_t^{k+1}(x))}{|v_i - x|}. \tag{3.32}$$

We shall first prove that

$$M := \#\{1 \leq k \leq \widehat{S}_i - S_i : f_t^k(v_i) \in \widetilde{B}(\varepsilon')\} \leq \rho_2 d_i (C + a + 1)^{-1} < n. \tag{3.33}$$

and

$$\sum_{k=0}^{\widehat{S}_i-S_i} q_{\varepsilon'}(f_t^{k+1}(c_i)) \le (C+a+1)M \le \rho_2 d_i. \tag{3.34}$$

Indeed, $q_{\varepsilon'}(z) \le q_\varepsilon(z) + a + 1$ holds for each $z \in [0,1]$, thus

$$t \in X_{n,\varepsilon}(C) \subset X_{n,\varepsilon'}(C+a+1). \tag{3.35}$$

Therefore (3.34) will follow once we prove (3.33). Let $T_1 < T_2 < \cdots$ be all the positive integers such that $f_t^{T_k+1}(c_i) \in \widetilde{B}(\varepsilon')$ and p_k be the critical point of f which is closest to $f_t^{T_k+1}(c_i)$. Then for each $1 \le m < n$, by Lemma 3.1, we have

$$|Df_t^{T_m+1}(f_t(c_i))| \ge \left(\Lambda(\varepsilon')e^{-\ell_{max}(C+a+1)}\right)^m, \tag{3.36}$$

hence

$$\begin{aligned} \mathrm{A}(v_i, f_t, T_m+1) &\ge \frac{|Df_t^{T_m}(f_t(c_i))|}{|f_t^{T_m+1}(c_i) - p_m|} = \frac{|Df_t^{T_m+1}(f_t(c_i))|}{|Df_t(f_t^{T_m+1}(c_i))||f_t^{T_m+1}(c_i) - p_m|} \\ &\ge |Df_t^{T_m+1}(f_t(c_i))|(\varepsilon')^{-1} \ge \left(\Lambda(\varepsilon')e^{-\ell_{max}(C+a+1)}\right)^m (\varepsilon')^{-1}. \end{aligned}$$

On the other hand, by Lemma 3.10, we have

$$\mathrm{A}(v_i, f_t, \widehat{S}_i - S_i) \asymp \mathrm{A}(x, f_t, \widehat{S}_i - S_i) \le \theta_0 e^{(d_i-1)\ell(c_i)}\varepsilon^{-1}.$$

Hence, there exists a constant $C_1 > 0$ such that

$$\left(\Lambda(\varepsilon)e^{-\ell_{max}(C+a+1)}\right)^M (\varepsilon')^{-1} \le C_1\theta_0 e^{(d_i-1)\ell(c_i)}\varepsilon^{-1}.$$

The inequality (3.33) follows, provided that $\varepsilon > 0$ is small enough.

Let us now prove (3.29). Indeed, by (3.31), for each $0 \le k < \widehat{S}_i - S_i$, we have $|Df_t(f_t^k(v_i))| \le e^2|Df_t(f_t^k(x))|$, so $q_{\varepsilon'}(f_t^k(v_i)) \ge q_\varepsilon(f_t^k(x))$. Thus

$$\sum_{i<k<j} d_k \le \sum_{k=0}^{\widehat{S}_i-S_i} q_{\varepsilon'}(f_t^{k+1}(c_i)) \le \rho_2 d_i, \tag{3.37}$$

which implies (3.29) since $\rho_2 < \rho_1$.

To obtain (3.30) it suffices to prove the following two inequalities:

$$|Df_t^{S_j-S_i-1}(x)| \geq \Lambda_1(\varepsilon)^{j-i} \left(D_{c_j}(\varepsilon)\right)^{-1}, \tag{3.38}$$

and

$$|Df_t^{S_j-S_i-1}(x)| \geq \kappa \exp\left(\ell(c_i)d_i - \rho_2 \ell_{max} d_i\right) \left(D_{c_j}(\varepsilon)\right)^{-1}, \tag{3.39}$$

where $\Lambda_1(\varepsilon) \to \infty$ as $\varepsilon \to 0$ and $\kappa > 0$ is a constant.

Indeed, combining these two inequalities, we obtain

$$U := |Df_t^{S_j-S_i-1}(x)| D_{c_j}(\varepsilon) \geq 3^{j-i} \exp\left(\ell(c_i)d_i - \rho_1 d_i + \ell_{\max}\right).$$

Since

$$\frac{P_j^{(c)}}{P_i^{(c)}} = U \frac{|Df_t(y)||y-c_i|}{|f_t^{S_j+1}(c) - c_j| D_{c_j}(\varepsilon)} \geq U \exp\left(-\ell(c_i)d_i + d_j - 1\right),$$

the inequality (3.30) follows.

Let us prove (3.38). Applying Proposition 2.1 (i), we obtain

$$|Df_t^{S_j-S_{j-1}-1}(f_t^{S_{j-1}-S_i}(x))| \geq \Lambda(\varepsilon)/D_{c_j}(\varepsilon).$$

Thus (3.38) holds with $\Lambda_1(\varepsilon) = \Lambda(\varepsilon)$ if $j = i+1$. When $j > i+1$, $S_{j-1} - S_i$ is of the form $T_m + 1$ for some $j - i - 1 \leq m \leq M < n$, so combining (3.36) with the last inequality, we obtain that (3.38) holds with a suitable choice of $\Lambda_1(\varepsilon)$.

Finally let us prove (3.39). We may certainly assume $(\ell_{max} - 1)\rho_2 d_i \geq 2$. Let

$$A_k = \frac{|Df_t^{S_k-S_i-1}(x)|}{\operatorname{dist}(f_t^{S_k-S_i-1}(x), \mathcal{C})}, \text{ and } A'_k = \frac{|Df_t^{S_k-S_i-1}(x)|}{|\widetilde{B}(c_k;\varepsilon)|}$$

for $i < k \leq j$. Clearly, $A_k \geq A'_k$. By Proposition 2.1 (i), we have

$$\frac{A'_j}{A_k} = |Df_t^{S_j-S_k}(f_t^{S_k+1}(c))| \frac{\operatorname{dist}(f_t^{S_k+1}(c), \mathcal{C})}{|\widetilde{B}(c_j;\varepsilon)|} \geq \Lambda(\varepsilon)^{j-k} \exp\left(-\ell_{max} \sum_{k \leq l < j} d_l\right),$$

which, by (3.37), implies

$$\frac{A'_j}{A_k} \geq \Lambda(\varepsilon)^{j-k} e^{-\rho_2 \ell_{max} d_i}. \tag{3.40}$$

Let $\theta = \theta_0/(2e^{\ell_{max}})$. We distinguish two cases.

Case 1. Assume $\mathrm{A}(x,t,S_j-S_i-1)\geq \theta e^{d_i\cdot\ell(c_i)}\varepsilon^{-1}$. Then by Lemma 3.9, we have

$$\sum_{k=i+1}^{j-1} A_k + A'_j \geq \frac{1}{1+\kappa(\varepsilon)}\mathrm{A}(x,f_t,S_j-S_i-1)\geq \theta e^{d_i\cdot\ell(c_i)}(2\varepsilon)^{-1}.$$

Together with (3.40), this implies $A'_j\geq \theta\exp(\ell(c_i)d_i-\rho_2\ell_{max}d_i)(4\varepsilon)^{-1}$, provided that $\varepsilon>0$ is small enough. Thus (3.39) holds in this case.

Case 2. Assume $\mathrm{A}(x,f_t,S_j-S_i-1)<\theta e^{\ell(c_i)\cdot d_i}\varepsilon^{-1}$. In particular we have $S_j-1\leq \widehat{S}_i$ which implies $\widehat{S}_i=S_j-1$. By maximality of $\widehat{S}_i$ we have

$$A_j = A(x,f_t,S_j-S_i)-A(x,f_t,\widehat{S}_i-S_i)\geq \theta e^{\ell(c_i)\cdot d_i}\varepsilon^{-1}.$$

So (3.39) holds if $d_j\leq \rho_2\ell_{max}d_i$. Assume $d_j>\rho_2\ell_{max}d_i$. By (3.34),

$$q_{\varepsilon'}(f_t^{S_j-S_i-1}(v_i))\leq \rho_2 d_i\leq \rho_2\ell_{max}d_i-2.$$

Thus there exists a constant $\kappa_1>0$ such that

$$\mathrm{dist}(f_t^{S_j-S_i-1}(v_i),f_t^{S_j-S_i-1}(x))\geq \kappa_1 e^{-\rho_2\ell_{max}d_i}|\widetilde{B}(c_j;\varepsilon)|$$

Thus, by (3.32),

$$|Df_t^{S_j-S_i-1}(x)|\geq e^{-1}\frac{\mathrm{dist}(f_t^{S_j-S_i-1}(v_i),f_t^{S_j-S_i-1}(x))}{|v_i-x|}\geq \frac{\kappa_1\exp(\ell(c_i)d_i-\rho_2\ell_{max}d_i)}{D_{c_j}(\varepsilon)}.$$

So the inequality (3.39) holds. □

Lemma 3.12. *An essential return time is a free return time.*

Proof. By definition, for any consecutive free return times $S_i<S_j$, we have $P_k<P_i$ for all $i<k<j$. So S_k is not an essential return time. The lemma follows. □

We say that a free return is *inessential* if it is not essential return.

Lemma 3.13. *Assume $t\in X_{n,\varepsilon}(C)$. Let $c\in\mathcal{C}$. If $1\leq i<j\leq n$ are such that $S_i^{(c)}<S_j^{(c)}$ are consecutive essential return times of $f_t(c)$ into $\widetilde{B}(\varepsilon)$, then*

$$\sum_{i<k<j} d_k^{(c)}\leq \rho d_i^{(c)}, \tag{3.41}$$

and

$$p_j^{(c)} \geq d_j^{(c)} - \rho d_i^{(c)}. \tag{3.42}$$

Moreover, if n_0 *is the largest integer in* $\{1, 2, \ldots, n\}$ *such that* $S_{n_0}^{(c)}$ *is an essential return time of* $f_t(c)$ *into* $\widetilde{B}(\varepsilon)$*, then*

$$\sum_{n_0 < k \leq n} d_k^{(c)} \leq \rho d_{n_0}^{(c)}. \tag{3.43}$$

Proof. By Lemma 3.12, $S_i^{(c)}$ and $S_j^{(c)}$ are both free return times. Let $i = k_0 < k_1 < \ldots < k_m = j$ be all the positive integers such that $S_{k_l}^{(c)}$ are free return times. Then by Lemma 3.11, for each $0 \leq l < m$, we have

$$\log \frac{P_{k_{l+1}}^{(c)}}{P_{k_l}^{(c)}} \geq d_{k_{l+1}}^{(c)} - \rho_1 d_{k_l}^{(c)} + (\log 3)(k_{l+1} - k_l), \tag{3.44}$$

and

$$\sum_{k_l < k < k_{l+1}} d_k^{(c)} \leq \rho_1 d_{k_l}^{(c)}. \tag{3.45}$$

Summing up both sides of (3.44) for $0 \leq l < m-1$, we obtain

$$\log \frac{P_{k_{m-1}}^{(c)}}{P_{k_0}^{(c)}} \geq d_{k_{m-1}}^{(c)} + (1-\rho_1) d_{k_{m-2}}^{(c)} + \cdots + (1-\rho_1) d_{k_1}^{(c)} - \rho_1 d_{k_0}^{(c)} + (\log 3)(k_{m-1} - k_0).$$

Since the left hand side is smaller than $(\log 3)(k_{m-1} - k_0)$, we obtain

$$d_{k_{m-1}}^{(c)} + (1-\rho_1) d_{k_{m-2}}^{(c)} + \cdots + (1-\rho_1) d_{k_1}^{(c)} \leq \rho_1 d_{k_0}^{(c)},$$

which, together with (3.45), implies

$$\sum_{i<k<j} d_k^{(c)} \leq \rho_1 d_i^{(c)} + (1+\rho_1)\rho_1(1-\rho_1)^{-1} d_i^{(c)} \leq \rho d_i^{(c)}.$$

This proves (3.41). Summing up both sides of (3.44) for $0 \leq l < m$,

$$\log \frac{P_j^{(c)}}{P_i^{(c)}} \geq d_j^{(c)} + (1-\rho_1) \sum_{l=1}^{m-1} d_{k_l}^{(c)} - \rho_1 d_{k_0}^{(c)} + (\log 3)(j-i) \geq d_j^{(c)} - \rho_1 d_i^{(c)} + (\log 3)(j-i),$$

which is equivalent to

$$P_i^{(c)} \leq e^{-(d_j^{(c)}-\rho_1 d_i^{(c)})} 3^{i-j} P_j^{(c)}. \tag{3.46}$$

Let us now prove that

$$P_k^{(c)} \leq 3^{k-i} P_i^{(c)} \text{ for any } 1 \leq k \leq j-1. \tag{3.47}$$

Indeed, for $1 \leq k < i$, this inequality follows from that fact that $S_i^{(c)}$ is an essential return times, while for $i < k < j$, it follows from the fact that $S_k^{(c)}$ is not an essential return time.

Combining (3.46) and (3.47), we obtain

$$\sum_{k=1}^{j-1} P_k^{(c)} \leq e^{-(d_j^{(c)}-\rho_1 d_i^{(c)})} 2^{-1} P_j^{(c)}.$$

By Lemma 3.9, we have

$$\mathrm{A}(f_t(c), f_t, S_j) \leq (1+\kappa(\varepsilon)) \sum_{k=1}^{j-1} P_k^{(c)} + \kappa(\varepsilon) \frac{|Df_t^{S_j}(f_t(c))|}{|\widetilde{B}(c_j;\varepsilon)|},$$

where c_j is the critical point of f which is closest to $f_t^{S_j^{(c)}+1}(c)$ and $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$. So when $\varepsilon > 0$ is small, we obtain

$$\mathrm{A}(f_t(c), f_t, S_j) \leq e^{-(d_j - \rho_1 d_i)} P_j^{(c)}.$$

The inequality (3.42) follows.

The inequality (3.43) can be proved in a similar way. □

Proof of Proposition 3.8. (i) By definition, there exists c such that $\sum_{k=1}^n d_k^{(c)} \geq Cn$. Let $i_1 < i_2 < \cdots < i_m$ be all the integers in $\{1, 2, \ldots, n\}$ such that S_{i_j} is an essential return times of $f_t(c)$ into $\widetilde{B}(\varepsilon)$. Then $i_1 = 1$. For convenience of notations, we regard $i_0 = 0$ and $d_{i_0}^{(c)} = 0$.

By (3.41) and (3.43) in Lemma 3.13, we have

$$\sum_{j=1}^m d_{i_j}^{(c)} \geq (1-\rho) \sum_{j=1}^n d_j^{(c)} \geq (1-\rho) Cn.$$

Thus
$$\sum_{j=1}^{m}\left(d_{i_j}^{(c)}-\rho d_{i_{j-1}}^{(c)}\right)\geq(1-\rho)^2Cn.$$

By (3.42) in Lemma 3.13, for each $2\leq j\leq m$ we have
$$p_{i_j}^{(c)}\geq d_{i_j}^{(c)}-\rho d_{i_{j-1}}^{(c)}.$$

By Lemma 3.9, this estimate is also true for $j=1$. Thus
$$\widetilde{p}_{i_j}^{(c)}\geq d_{i_j}^{(c)}-\rho d_{i_{j-1}}^{(c)}$$

holds for all $j=1,2,\ldots,m$, which implies
$$\sum_{j=1}^{m}\widetilde{p}_{i_j}^{(c)}\geq\sum_{j=1}^{m}\left(d_{i_j}^{(c)}-\rho d_{i_{j-1}}^{(c)}\right)\geq(1-\rho)^2Cn.$$

Consequently,
$$\sum_{k\in\widehat{\mathcal{T}}_{\mathrm{ess}}^{(c)}(C_0,t;\varepsilon),k\leq n}\widetilde{p}_k^{(c)}\geq\sum_{j=1}^{m}\widetilde{p}_{i_j}^{(c)}-C_0n\geq(1-\rho)^2Cn-C_0n=(\gamma C-C_0)n.$$

(ii) By definition, there exists $c,c'\in\mathcal{C}$ such that $\mathrm{dist}(f_t^{m+1}(c),c')\leq\varepsilon^{1/\ell(c')}(m+1)^{-\tau}$. So there exists $n\geq 1$ such that $m=S_n^{(c)}$ and $d_n^{(c)}\geq\tau\log(S_n^{(c)}+1)$. Since $d_k^{(c)}<d_n^{(c)}$ holds for each $1\leq k<n$, by (3.41) in Lemma 3.13 it follows that n is an essential return time of $f_t(c)$ into $\widetilde{B}(\varepsilon)$ and hence $p_n^{(c)}\geq(1-\rho)d_n^{(c)}>C_0$. The statement is proved. □

3.3 Harvest in the parameter space

In this section, we transfer the estimates in phase space to the parameter space and prove the Reduced Theorem A. The phase and parameter spaces are related through the maps $\xi_n^{(c)}(t)=f_t^{n+1}(c)$. In § 3.3.1, we define parameter boxes. In § 3.3.3, we prove the Reduced Theorem A by showing that the bad parameters are contained in certain families of parameter boxes with large total depth. Proposition 3.3 will be used to construct the parameter boxes and Proposition 3.8 will be used to estimate the total depth. The parameter boxes which we use are always mapped into $\widetilde{B}(\varepsilon)$

and they form special families of balls. In § 3.3.2, an abstract lemma about sets of points lying deeply in a special family of balls is proved.

3.3.1 Parameter boxes

Recall that

$$\frac{\partial_t \xi_n^{(c)}(t)}{Df_t^n(f_t(c))} = \sum_{j=0}^{n} \frac{\partial_t F(f_t^j(c), t)}{Df_t^j(f_t(c))} =: M_n^{(c)}(t).$$

Definition 3.2. Given $m \geq 0$, $\lambda > 1$ and $c \in \mathcal{C}$, we say that a ball $B(t_0, r)$ in the parameter space is a *λ-bounded c-parameter box of order m* if the following hold:

- $\xi_m^{(c)} : B(t_0, r) \to [0, 1]$ is a diffeomorphism onto its image such that

$$\sup_{t_1, t_2 \in B(t_0, r)} \frac{\partial_t \xi_m^{(c)}(t_1)}{\partial_t \xi_m^{(c)}(t_2)} \leq \lambda.$$

- For any $t \in B(t_0, r)$, we have

$$\lambda^{-1} |a^{(c)}| \leq \left| M_m^{(c)}(t) \right| \leq \lambda |a^{(c)}|.$$

- for each $k = 0, 1, \ldots, m$, we have

$$\sup_{t, s \in B(t_0, r)} \frac{|Df_t^k(f_t(c))|}{|Df_s^k(f_s(c))|} \leq \lambda.$$

The goal of this section is to provide an estimate of the size of a parameter box centered at a given parameter t_0.

Proposition 3.14. *Given $\lambda > 1$, there exist $\theta > 0$ and $N \geq 1$ such that the following holds. Let $|t_0| \leq \theta$, $c \in \mathcal{C}$ and $m > N$ be such that*

$$\sum_{i=0}^{m} |Df_{t_0}^i(f_{t_0}(c))|^{-1} \leq W^{(c)} + \theta,$$

then putting

$$r = \theta / A(f_{t_0}(c), f_{t_0}, m),$$

$B(t_0, r)$ is a λ-bounded c-parameter box of order m.

Write $D_n^{(c)}(t) = Df_t^n(f_t(c))$.

Lemma 3.15. *Given $\lambda > 1$ there exist $\eta = \eta(\lambda) > 0$ and an integer $N = N(\lambda) \geq 1$ such that the following holds. Let $t \in [-\eta, \eta]$, $c \in \mathcal{C}$ and let $m > N$ be a positive integer. Assume*

$$\sum_{i=0}^{m} |D_i^{(c)}(t)|^{-1} \leq W^{(c)} + \eta. \tag{3.48}$$

Then

$$\lambda^{-1}|a^{(c)}| < |M_m^{(c)}(t)| < \lambda|a^{(c)}|.$$

Proof. Take $\delta > 0$ small. Let N be large such that

$$\sum_{i=0}^{N} |Df^i(f(c))|^{-1} > W^{(c)} - \delta, \text{ and } \left|M_N^{(c)}(0) - a^{(c)}\right| < \delta.$$

By continuity, there exists $\eta_0 > 0$ such that for any $t \in [-\eta_0, \eta_0]$, we have

$$\sum_{i=0}^{N} |D_i^{(c)}(t)|^{-1} > W^{(c)} - \delta, \text{ and } \left|M_N^{(c)}(t) - a^{(c)}\right| < \delta.$$

Now let $\eta = \min(\delta, \eta_0)$. If (3.48) holds, then we have

$$\sum_{i=N+1}^{m} |D_i^{(c)}(t)|^{-1} < \delta + \eta \leq 2\delta.$$

Since $|\partial_t F| \leq 1$, it follows that

$$\left|M_m^{(c)}(t) - a^{(c)}\right| \leq \left|M_N^{(c)}(t) - a^{(c)}\right| + \sum_{i=N+1}^{m} |D_i^{(c)}(t)|^{-1} < 3\delta.$$

The desired inequality follows since $\min_{c\in\mathcal{C}} |a^{(c)}| > 0$. $\square$

Proof of Proposition 3.14. Fix $\lambda > 1$. Let $\lambda_0 = \lambda^{1/4}$ and let $\eta = \eta(\lambda_0), N = N(\lambda_0)$ be given by Lemma 3.15. Let $\theta \in (0, \eta/2)$ and $\lambda_1 \in (1, \lambda_0)$ be such that

$$\lambda_1(W^{(c)} + \theta) \leq W^{(c)} + \eta,$$

holds for each $c \in \mathcal{C}$.

Now let t_0, c, m be as in the assumption of this proposition. Then by continuity, there exists a maximal $r_0 \in (0, \theta]$ such that for each $t \in \overline{B(t_0, r_0)}$ and any $1 \leq j \leq m$, we have

$$\frac{1}{\lambda_1} \leq \frac{D_j^{(c)}(t)}{D_j^{(c)}(t_0)} \leq \lambda_1. \tag{3.49}$$

So

$$\sum_{i=0}^{m} |D_i^{(c)}(t)|^{-1} \leq \lambda_1 \sum_{i=0}^{m} |D_i^{(c)}(t_0)|^{-1} \leq \lambda_1 (W^{(c)} + \theta) \leq W^{(c)} + \eta,$$

which implies by Lemma 3.15 that

$$\lambda_0^{-1} |a^{(c)}| \leq |M_m^{(c)}(t)| \leq \lambda_0 |a^{(c)}|.$$

It follows that $B(t_0, r_0)$ is a λ-bounded c-parameter box of order m. So it suffices to prove that $\theta_0 := r_0 \cdot \mathrm{A}(f_{t_0}(c), t_0, m)$ is bounded away from zero. To this end, we only need to show that there exists a constant $C > 0$ such that for each $1 \leq j \leq m$,

$$\left| \log \frac{D_j^{(c)}(t)}{D_j^{(c)}(t_0)} \right| \leq C\theta_0. \tag{3.50}$$

Indeed, if $r_0 = \theta$ then $\theta_0 \geq r_0 = \theta$, and if $r_0 < \theta$, then by maximality of r_0, there exists $t_1 \in \overline{B(t_0, r_0)}$ and $j \in \{1, 2, \ldots, m\}$ such that

$$\text{either } D_j^{(c)}(t_1) = \lambda_1 D_j^{(c)}(t_0) \text{ or } D_j^{(c)}(t_0) = \lambda_1 D_j^{(c)}(t_1). \tag{3.51}$$

Thus (3.50) implies that $\theta_0 \geq \min(\log \lambda_1 / C, \theta)$.

Let us prove (3.50). First note that there exists $C_1 > 0$ such that for each $1 \leq j \leq m$ and any $t \in \overline{B(t_0, r_0)}$, we have $|M_j^{(c)}(t)| \leq C_1$, so

$$|\partial_t \xi_j^{(c)}(t)| = |M_j^{(c)}(t) D_j^{(c)}(t)| \leq C_1 |D_j^{(c)}(t)|.$$

Since F is a normalized regular family, there exists a constant $C_2 > 0$ such that

$$|\partial_t \partial_x F(x, t)| \leq C_2 |Df_t(x)|,$$

for all (x, t). (Indeed, for x close to $\mathcal{C}$ the left hand side of this inequality is zero.)

By (3.49), for each $0 \le i < m$,

$$\frac{|Df_t(\xi_i^{(c)}(t))|}{|Df_{t_0}(\xi_i^{(c)}(t_0))|} = \frac{|D_{i+1}^{(c)}(t)|}{|D_{i+1}^{(c)}(t_0)|}\frac{|D_i^{(c)}(t_0)|}{|D_i^{(c)}(t)|} \in [\lambda^{-1}, \lambda].$$

By non-flatness of the critical points, it follows that there exist constants C_3 and C_4 such that

$$\frac{|D^2 f_t(\xi_i^{(c)}(t))|}{|Df_t(\xi_i^{(c)}(t))|} \le C_3 \frac{|D^2 f_{t_0}(\xi_i^{(c)}(t_0))|}{|Df_{t_0}(\xi_i^{(c)}(t_0))|} \le \frac{C_4}{\operatorname{dist}(\xi_i^{(c)}(t_0), \mathcal{C})}.$$

Since $\operatorname{dist}(\xi_i^{(c)}(t_0), \mathcal{C}) \le 1$, and $|D_i^{(c)}(t)| \ge \lambda_1^{-1}|D_i^{(c)}(t_0)|$ is bounded away from zero, there exists a constant $C_5 > 0$ such that

$$\begin{aligned}\left|\partial_t Df_t(\xi_i^{(c)}(t))\right| &= \left|\partial_t\partial_x F(\xi_i^{(c)}(t), t) + D^2 f_t(\xi_i^{(c)}(t)) \cdot \partial_t \xi_i^{(c)}(t)\right| \\ &\le C_2|Df_t(\xi_i^{(c)}(t))| + C_4 C_1 |Df_t(\xi_i^{(c)}(t))| \frac{|D_i^{(c)}(t)|}{\operatorname{dist}(\xi_i^{(c)}(t_0), \mathcal{C})} \\ &\le C_5 |Df_t(\xi_i^{(c)}(t))| \frac{|D_i^{(c)}(t)|}{\operatorname{dist}(\xi_i^{(c)}(t_0), \mathcal{C})}.\end{aligned}$$

Thus

$$\begin{aligned}\left|\partial_t \log|D_j^{(c)}(t)|\right| &= \left|\sum_{i=0}^{j-1} \frac{\partial_t Df_t(\xi_i^{(c)}(t))/dt}{Df_t(\xi_i^{(c)}(t))}\right| \\ &\le C_5 \sum_{i=0}^{j-1} \frac{|D_i^{(c)}(t)|}{\operatorname{dist}(\xi_i^{(c)}(t_0), \mathcal{C})} \le C\mathrm{A}(f_{t_0}(c), f_{t_0}, j),\end{aligned}$$

where $C = C_5\lambda_1$. Since

$$\log \frac{D_j^{(c)}(t)}{D_j^{(c)}(t_0)} = \int_{t_0}^{t} \partial_t \log|D_j^{(c)}(t)|,$$

the inequality (3.50) follows. □

3.3.2 Special family of balls

In this subsection, we will introduce the notion of *special family* that provides a tool to estimate the bad parameters and that is a variation of the argument in [36].

Given $B = B(a_0, r)$ and $x \in \mathbb{R}$, we define

$$\operatorname{dep}(x|B) = \begin{cases} \inf\{k \in \mathbb{N} : |x - a_0| \geq e^{-k} r\}, & \text{if } |x - a_0| < e^{-2} r; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, for each $k \in \mathbb{Z}$, let

$$B^{(k)} = B(a_0, e^{-k} r). \tag{3.52}$$

A finite family $\mathcal{M} = \{B_i = B(a_i, r_i)\}_{i \in \mathcal{I}}$ is called *special* if the following holds: For any $i, j \in \mathcal{I}$, if $a_i \in B_j^{(1)}$ then there exists $k = k(i,j) \geq 1$ such that $B_i \subset B_j^{(k-1)} \setminus B_j^{(k+1)}$. In particular, the centers $a_i, i \in \mathcal{I}$ are pairwise distinct. See Figure 3.1.

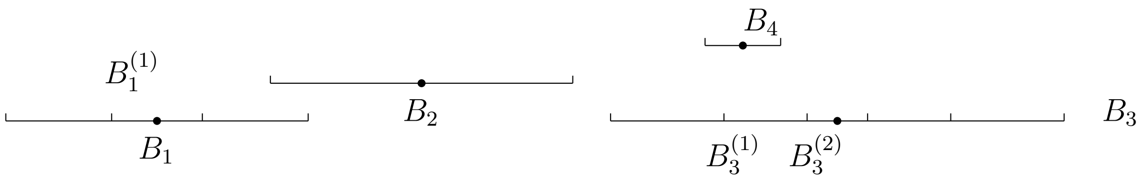

Figure 3.1: A special family

Given a special family as above, define a partition of $\mathcal{I}$ into sets $\mathcal{I}_0, \mathcal{I}_1, \ldots,$ inductively as follows. Put

$$\mathcal{I}_0 = \{i \in \mathcal{I} : \text{ for any } j \in \mathcal{I}, j \neq i, \text{ we have } a_i \notin B_j^{(1)}\},$$

and for each $k \geq 1$, let

$$\mathcal{I}_k = \left\{ i \in \mathcal{I} \setminus \bigcup_{m=0}^{k-1} \mathcal{I}_m : \text{ for any } j \in \mathcal{I} \setminus \bigcup_{m=0}^{k-1} \mathcal{I}_m, j \neq i, \text{ we have } a_i \notin B_j^{(1)} \right\}.$$

The minimal integer $n \geq 0$ for which $\mathcal{I}_n = \emptyset$, if any, is called the *height* of $\mathcal{M}$. The support of $\mathcal{M}$ is defined as the union of all the elements of $\mathcal{M}$. See Figure 3.2.

We shall use the next lemma to estimate measure of sets of bad parameters.

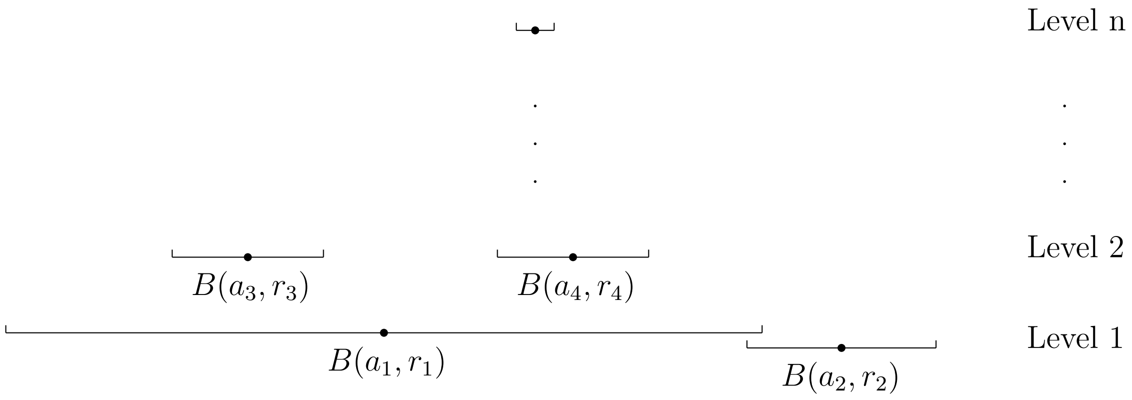

Figure 3.2: The height of a special family

Lemma 3.16. *For each $0 < \kappa < 1$ there exists $K = K(\kappa) > 1$ such that if $\mathcal{M} = \{B_i\}_{i\in\mathcal{I}}$ is a special family of height at most n and*

$$X_{\mathcal{M}}(N) = \left\{ x \in supp(\mathcal{M}) : \sum_{i\in\mathcal{I}} dep(x|B_i) \geq N \right\}, N = 0, 1, \ldots,$$

then

$$|X_{\mathcal{M}}(N)| \leq K^n e^{-(1-\kappa)N} \left|supp(\mathcal{M})\right|.$$

Proof. Fix $0 < \kappa < 1$ and let $K = K(\kappa) = e^5/(1 - e^{-\kappa})$. We shall prove the lemma by induction on the height n. We take the trivial case $n = 0$ for the starting step. Now let n_0 be a positive integer and assume that the lemma holds for $n < n_0$. Let us consider the case $n = n_0$. Let $\mathcal{I}_0$ be defined as above, and let $\mathcal{I}' = \mathcal{I} \setminus \mathcal{I}_0$. Let

$$q_0(x) = \sum_{i\in\mathcal{I}_0} \mathrm{dep}(x|B_i), \ q'(x) = \sum_{i'\in\mathcal{I}'} \mathrm{dep}(x|B_{i'}).$$

For each $q_0 \geq 0$, and $q' \geq 0$, let $V(q_0) = \{x \in \mathrm{supp}(\mathcal{M}) : q_0(x) = q_0\}$ and $U(q_0, q') = \{x \in V(q_0) : q'(x) \geq q'\}$. Let us prove that

$$|U(q_0, q')| \leq e^{-q_0+5} K^{n-1} e^{-(1-\kappa)q'} \left|\mathrm{supp}(\mathcal{M})\right|. \tag{3.53}$$

To this end, we first note that the balls $B_i^{(2)}$, $i \in \mathcal{I}_0$, are pairwise disjoint. Thus

$$V(q_0) \subset \bigcup_{i\in\mathcal{I}_0} B_i^{(q_0-1)}, \tag{3.54}$$

and for each $k = 0, 1, \dots,$ we have

$$\sum_{i \in \mathcal{I}_0} |B_i^{(k)}| \le e^{-k+2} |\mathrm{supp}(\mathcal{M})|. \tag{3.55}$$

In particular, $|V(q_0)| \le e^{-q_0+3}|\mathrm{supp}(\mathcal{M})|$, so the inequality (3.53) holds when $q' = 0$. Assume now that $q' > 0$ and let

$$\mathcal{M}'_{q_0} = \{B_{i'} : i' \in \mathcal{I}', B_{i'}^{(2)} \cap V(q_0) \neq \emptyset\}.$$

Then $\mathcal{M}'_{q_0}$ is a special family of height $< n$ and $U(q_0, q') \subset X_{\mathcal{M}'_{q_0}}(q')$. By the induction hypothesis, we have

$$|U(q_0, q')| \le \left| X_{\mathcal{M}'_{q_0}}(q') \right| \le K^{n-1} e^{-(1-\kappa)q'} \left| \mathrm{supp}(\mathcal{M}'_{q_0}) \right|. \tag{3.56}$$

Next, let us show

$$\mathrm{supp}(\mathcal{M}'_{q_0}) \subset \bigcup_{i \in \mathcal{I}_0} B_i^{(q_0-3)}. \tag{3.57}$$

In fact, since $\mathcal{M}'_{q_0} \subset \mathcal{M}$, (3.57) holds when $q_0 \le 3$. Assume $q_0 > 3$. By (3.54), for each $B_{i'} \in \mathcal{M}'_{q_0}$ there exists $i \in \mathcal{I}_0$ such that $B_{i'}^{(2)} \cap B_i^{(q_0-1)} \neq \emptyset$. By definition of special family, we have $B_{i'} \subset B_i^{(q_0-3)}$. Thus (3.57) holds. By (3.55), it follows that

$$\left| \mathrm{supp}(\mathcal{M}'_{q_0}) \right| \le e^{-q_0+5} |\mathrm{supp}(\mathcal{M})|,$$

which, together with (3.56), implies (3.53).

Now let us complete the induction step. Fix $N \ge 0$ and for each $q_0 \ge 0$, let $q'_0 = \max(N - q_0, 0)$. So $q_0 + q'_0 \ge N$. Since

$$X_{\mathcal{M}}(N) \subset \bigcup_{q_0=0}^{\infty} U(q_0, q'_0),$$

by (3.53), we obtain

$$\begin{aligned} |X_{\mathcal{M}}(N)| \le \sum_{q_0=0}^{\infty} |U(q_0, q'_0)| &\le K^{n-1} e^5 |\mathrm{supp}(\mathcal{M})| \sum_{q_0=0}^{\infty} e^{-\kappa q_0} e^{-(1-\kappa)N} \\ &= K^n e^{-(1-\kappa)N} |\mathrm{supp}(\mathcal{M})|. \end{aligned}$$

This completes the proof. $\square$

3.3.3 Proof of the Reduced Theorem A

For $c \in \mathcal{C}$ and $m \geq 0$, let $\mathscr{C}_m^{(c)}$ denote the set of parameters $t \in [0,1]$ for which the following hold:

- $f_t^{m+1}(c) \in \mathcal{C}$;
- $f_t^j(c) \cap \mathcal{C} = \emptyset$ for all $1 \leq j \leq m$.

A c-parameter-box $B(t,r)$ of order m is called *pre-critical* if $t \in \mathscr{C}_m^{(c)}$.

For $m \geq 0$, $c \in \mathcal{C}$, $t_* \in \mathscr{C}_m^{(c)}$ and $\lambda > 1$, let $r_\lambda(t_*, \varepsilon)$ be the maximal number r which satisfy the following properties:

(i) $r \in (0, \varepsilon]$ and $\xi_m^{(c)}(B(t_*, r)) \subset \widetilde{B}(\varepsilon)$;

(ii) $B(t_*, r)$ is a λ-bounded pre-critical c-parameter boxes of order m.

Given a positive integer n, let

$$\mathcal{M}_{n,\lambda}^{(c)}(\varepsilon) = \left\{ B(t_*, r_\lambda(t_*, \varepsilon)) \;\middle|\; \begin{array}{c} t_* \in \mathscr{C}_m^{(c)} \text{ for some } m \geq 0 \text{ and} \\ \text{there exists } t \in B(t_*, r_\lambda(t_*, \varepsilon)) \text{ such that} \\ \#\{0 \leq j \leq m : f_t^{j+1}(c) \in \widetilde{B}(\varepsilon)\} \leq n. \end{array} \right\}.$$

See Figure 3.3.

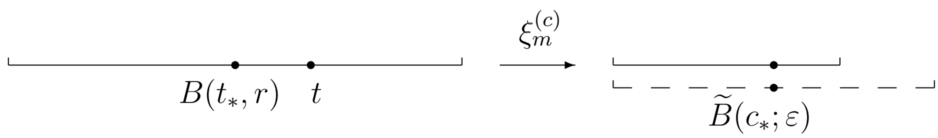

Figure 3.3: An element of $\mathcal{M}_{n,\lambda}^{(c)}(\varepsilon)$, where $\#\{0 \leq j \leq m : f_t^{j+1}(c) \in \widetilde{B}(\varepsilon)\} \leq n$.

Lemma 3.17. *There exists $\lambda > 1$ such that for each $c \in \mathcal{C}$, each $n \geq 1$ and each $\varepsilon > 0$ small, $\mathcal{M}_{n,\lambda}^{(c)}(\varepsilon)$ is a special family of height at most n.*

Proof. Assume that $\lambda > 1$ is very close to 1. To prove that $\mathcal{M}_{n,\lambda}^{(c)}(\varepsilon)$ is special, let $B_i = B(t_i, r_i)$, $i = 1, 2$, be distinct parameter boxes in $\mathcal{M}_{n,\lambda}^{(c)}(\varepsilon)$, of order m_i, such that $t_1 \in B_2^{(1)}$. We need to prove that $|B_1|/|t_1 - t_2|$ is small. Let $c_1, c_2 \in \mathcal{C}$ be such that $\xi^{(c)}_{m_i}(B_i) \subset \widetilde{B}(c_i;\varepsilon)$. Since $f^{j+1}_{t_1}(c) \notin \mathcal{C}$ for each $0 \le j \le m_2$, we have $m_1 > m_2$. By the bounded distortion property of $\xi^{(c)}_{m_2}|B_2$, it suffices to show that

$$\sup_{t\in B_1} \frac{|\xi^{(c)}_{m_2}(t) - \xi^{(c)}_{m_2}(t_1)|}{|\xi^{(c)}_{m_2}(t_1) - c_2|}$$

is sufficiently small. This is clear: for each $t \in B_1$, and for each $0 \le k \le m_2+1 \le m_1$,

$$\lambda^{-1}|Df^k_{t_1}(f_{t_1}(c))| \le |Df^k_t(f_t(c))| \le \lambda|Df^k_{t_1}(f_{t_1}(c))|,$$

hence $\lambda^{-1}|Df_{t_1}(\xi^{(c)}_{m_2}(t_1))| \le |Df_t(\xi^{(c)}_{m_2}(t))| \le \lambda|Df_{t_1}(\xi^{(c)}_{m_2}(t_1))|$, so the statement follows from the local behavior of f near c_2.

Let us prove that the height of $\mathcal{M}^{(c)}_{n,\lambda}(\varepsilon)$ does not exceed n. Otherwise, there would exist $B_j \in \mathcal{M}^{(c)}_{n,\lambda}(\varepsilon)$, $0 \le j \le n$, such that $B_n \subsetneq B_{n-1} \subsetneq \cdots \subsetneq B_0$. Let m_j be the order of B_j. Then as above, we would have $m_0 < m_1 < \cdots < m_n$. Then for $t \in B_n$, $\{0 \le j \le m_n : f^{j+1}_t(c) \in \widetilde{B}(\varepsilon)\} \supset \{m_0, m_1, \ldots, m_n\}$ would contain at least $n+1$ elements, a contradiction. □

Now we fix a constant $\lambda > 1$ so that the conclusion of Lemma 3.17 holds. As before, we use $S^{(c)}_j(t;\varepsilon)$ to denote the j-th return time of $f_t(c)$ into $\widetilde{B}(\varepsilon)$. Then by Proposition 3.14, we have

Lemma 3.18. *There exists a constant $C_0 > 0$ such that for any $C > 0$ the following holds provided that $\varepsilon > 0$ is small enough. For $t \in X_{n,\varepsilon}(C)$, $c \in \mathcal{C}$ and $1 \le j \le n$, if $\widetilde{p}^{(c)}_j(t;\varepsilon) > C_0$, then there is a pre-critical c-parameter box $B(t_*, r)$ of order $S^{(c)}_j(t;\varepsilon)$ in $\mathcal{M}^{(c)}_{n,\lambda}(\varepsilon)$ such that*

$$dep(t|B(t_*, r)) \ge \widetilde{p}^{(c)}_j(t;\varepsilon) - C_0.$$

Proof. Fix C. By Corollary 3.4 and Proposition 3.14, provided that $\varepsilon > 0$ is small enough, there is a λ-bounded c-parameter box $B(t, r_0)$ of order $S_j := S^{(c)}_j(t,\varepsilon)$, with $r_0\mathrm{A}(f_t(c), t, S_j) = \theta$, where $\theta > 0$ is a constant independent of C. Let c' be the critical point such that $f^{S_j+1}_t(c) \in \widetilde{B}(c';\varepsilon)$, $p_j = p^{(c)}_j(t,\varepsilon)$ and $d_j = d^{(c)}_j(t,\varepsilon)$. Assume that p_j and d_j are large. Since $|\partial_t\xi^{(c)}_{S_j}(t)|\cdot r_0 \asymp |Df^{S_j}_t(f_t(c))|\cdot r_0 \asymp e^{p_j}\cdot|f^{S_j+1}_t(c) - c'|$, there exists a λ-parameter box $B(t_*, r_*) \subset B(t, r_0)$ of order S_j such that $r_* \asymp r_0$, $t_* \in \mathscr{C}^{(c)}_{S_j}$ and $\mathrm{dep}(t|B(t_*, r_*)) - p_j$ is bounded away from $-\infty$. See Figure 3.4. Let $r = r_\lambda(t_*, \varepsilon)$. Clearly, $B(t_*, r) \in \mathcal{M}^{(c)}_{n,\lambda}(\varepsilon)$. If $r \ge r_*$ then $\mathrm{dep}(t|B(t_*, r)) \ge \mathrm{dep}(t|B(t_*, r_*))$ and we are done. So assume $r < r_*$.

We claim that $\partial(\xi_{S_j}^{(c)}(B(t_*,r)))\cap\partial\widetilde{B}(c',\varepsilon)\neq\emptyset$. Otherwise we would have $r=\varepsilon$. Since $t\in X_{n,\varepsilon}(C)$, by Lemma 3.1, we would have that $|Df_t^{S_j}(f_t(c))|$ were much bigger than $(D_{c'}(\varepsilon))^{-1}$. It would then follow that $\xi_{S_j}^{(c)}(B(t_*,r))\supsetneqq\widetilde{B}(c',\varepsilon)$, contradicting the definition of r.

By the bounded distortion property of $\xi_{S_j}^{(c)}|B(t_*,r)$, it follows that $\xi_{S_j}^{(c)}(B(t_*,r))\supset\widetilde{B}(c',\varepsilon')$ holds for some $\varepsilon'\asymp\varepsilon$. Since $|f_t^{S_j+1}(c)-c'|\geq e^{-d_j}|\widetilde{B}(c';\varepsilon)|$, we conclude that $\mathrm{dep}(t|B(t_*,r))-d_j$ is bounded away from $-\infty$. The lemma is proved. $\square$

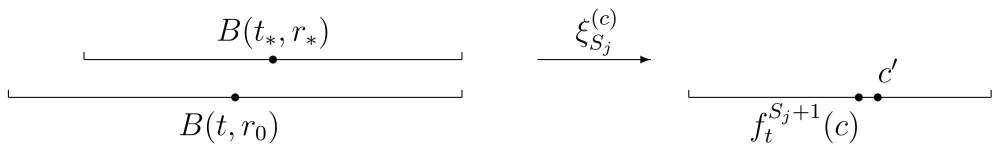

Figure 3.4: Lemma 3.18

Proof of the Reduced Theorem A. (i) Let $\lambda>1$ and C_0 be as above. We may assume $C>8C_0$. Consider $t\in X_{n,\varepsilon}(C)\setminus X_{n+1,\varepsilon}(C)$ with $\varepsilon>0$ small. By Proposition 3.8 (i) (taking $\gamma=1/2$) and Lemma 3.18, there exists $c\in\mathcal{C}$ such that

$$\sum_{B\in\mathcal{M}_{n,\lambda}^{(c)}(\varepsilon)}\mathrm{dep}(t|B)\geq\sum_{k\in\widehat{\mathcal{T}}_{\mathrm{ess}}^{(c)}(C_0,t;\varepsilon),k\leq n}\left(\widetilde{p}_k^{(c)}(t;\varepsilon)-C_0\right)\geq Cn/4. \tag{3.58}$$

By Lemma 3.17 and Lemma 3.16 (taking $\kappa=1/2$), it follows that

$$|X_{n,\varepsilon}(C)\setminus X_{n+1,\varepsilon}(C)|\leq\sum_{c\in\mathcal{C}}K^ne^{-Cn/8}\left|\mathrm{supp}(\mathcal{M}_{n,\lambda}^{(c)})\right|\leq(4\varepsilon\#\mathcal{C})\cdot K^ne^{-Cn/8},$$

where K is a constant.

(ii) It follows from Corollary 3.4 and Lemma 3.14.

(iii) Fix $C>0$, $\tau>\tau_0>1$ and $\kappa>0$. Let $\gamma\in(0,1)$ be a constant such that $\gamma^3\tau>\tau_0$. By Proposition 3.8(ii), for each $t\in Y_\varepsilon^m(C,\tau)\setminus Y_\varepsilon^{m+1}(C,\tau)$, there exist $c\in\mathcal{C}$ and $n\in\widehat{\mathcal{T}}_{\mathrm{ess}}^{(c)}(C_0,t;\varepsilon)$ such that $m=S_n^{(c)}(t;\varepsilon)$ and

$$p_n^{(c)}(t;\varepsilon)\geq\gamma d_n^{(c)}(t;\varepsilon)\geq\gamma\tau\log(S_n^{(c)}(t;\varepsilon)+1).$$

We may certainly assume that $Y_\varepsilon^m(C,\tau) \setminus Y_\varepsilon^{m+1}(C,\tau) \neq \emptyset$. Then

$$m \geq S(\varepsilon) := \inf\{S_1^{(c)}(t;\varepsilon) : |t| \leq \varepsilon, c \in \mathcal{C}\}$$

is large provided that $\varepsilon > 0$ is small. By Lemma 3.18 there exists $t_* \in \mathscr{C}_m^{(c)}$ such that

$$\mathrm{dep}(t|B(t_*, r_\lambda(t_*,\varepsilon))) \geq \gamma\tau\log(m+1) - C_0 \geq \gamma^2\tau\log(m+1).$$

Since these parameter boxes $B(t_*, r_\lambda(t_*,\varepsilon))$, $t_* \in \mathscr{C}_m^{(c)}$, are pairwise disjoint, it follows that

$$|Y_\varepsilon^m(C,\tau) \setminus Y_\varepsilon^{m+1}(C,\tau)| \leq C_1\varepsilon\#\mathcal{C}\,(m+1)^{-\gamma^3\tau}\,\varepsilon \leq \sigma\varepsilon(m+1)^{-\tau_0},$$

since m is large. $\square$

Chapter 4

Asymptotic distributions of the critical orbits

The aim of this chapter is to prove Theorem B. In § 4.1, we state Theorem B* and the Reduced Theorem B* from which we deduce Theorem B. As what I have done in § 3, the proof of the Reduced Theorem B* consists of two steps: in § 4.2 we "plough in the phase space" and in § 4.3 we "harvest in the parameter space".

4.1 Reduction

4.1.1 Theorem B*

In order to state a more precise version of our Theorem B, we need to introduce more definitions. Assume $f \in \mathcal{A}$ admits a unique acip μ. We say that f satisfies *the large deviation property* (abbreviated (LD)), if for any $\phi \in C^2([0,1], \mathbb{R})$ and any $\delta > 0$, there exist $C > 0$ and $\rho > 0$ such that

$$\left|\left\{x \in [0,1] : \left|\frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(x)) - \int \phi d\mu\right| > \delta\right\}\right| \le Ce^{-\rho n} \tag{4.1}$$

holds for each $n \in \mathbb{N}$.

Consider a C^1 family $f_t : [0,1] \to [0,1]$, $t \in [-1,1]$ and $z \in C^1([-1,1],[0,1])$. Define

$$\xi_n^{(z)}(t) := f_t^{n+1}(z(t)). \tag{4.2}$$

Note that if let $z(t) = c(t)$, this definition coincides the parameter map which is defined in §3. For each parameter $t_0 \in [-1, 1]$, we say that $z \in C^1([-1, 1], [0, 1])$ satisfies *the property (∗) at* t_0, if there exist $\lambda_1 \geq \lambda_2 > 1$, $\eta > 0$ and $\kappa \in (0, 1)$ such that the following hold for n large enough

- $|\partial_t \xi_n^{(z)}(t_0)| \geq \lambda_2^n$;
- there exists $r_n \leq \lambda_1^{-n}$ such that $\xi_n^{(z)}|J$ is a diffeomorphism and

  $$e^{-1}|\partial_t \xi_n^{(z)}(t)| \leq |\partial_t \xi_n^{(z)}(t_0)| \leq e|\partial_t \xi_n^{(z)}(t)|$$

  holds for any $t \in J$, where $J = [t_0 - r_n, t_0 + r_n] \cap [-1, 1]$;
- there exits $\kappa n \leq m \leq n$ such that

  $$|\xi_m^{(z)}([t_0 - r_m, t_0 + r_m])| \geq \eta.$$

We shall actually prove the following theorem B*, from which we can deduce Theorem B.

Theorem B*. Consider a normalized regular one-parameter family $f_t : [0, 1] \to [0, 1]$, $t \in [-1, 1]$. Fix $z \in C^1([-1, 1], [0, 1])$. Let Λ be the collection of parameters t with the following properties:

- $f_t \in \mathcal{A}$ and f_t satisfies the conditions (CE), (WR) and (NV$_t$);
- f_t admits a unique (hence ergodic) acip μ_t;
- f_t satisfies the property (LD);
- z satisfies the property (∗) at t.

Let $\Lambda_* \subset \Lambda$ be the collection of the parameters t for which the asymptotic distribution of the orbit of $z(t)$ for f_t exists and equals μ_t. Then we have $|\Lambda \setminus \Lambda_*| = 0$.

4.1.2 The (CE) and (WR) conditions

We are interested in the maps that satisfy the conditions (CE) and (WR). Moreover, to obtain meaningful results, we require that $\mathcal{C}(f) \cap \mathrm{PPer}(f) = \emptyset$, where $\mathrm{PPer}(f)$ denote the set of all pre-periodic points of f. The aim of this subsection is to prove the following proposition.

Proposition 4.1. *Consider a normalized regular family $(f_t)_{t\in[-1,1]}$. Let $\Delta_0 \subset \Delta$ be the collection of parameters t for which $\mathcal{C} \cap PPer(f_t) = \emptyset$ and f_t satisfies the conditions (CE) and (WR). Then $|\Delta \setminus \Delta_0| = 0$.*

Proof. Given $c \in \mathcal{C}$, $i \in \mathbb{N}$ and $j \in \mathbb{N}$, define

$$\Delta_{i,j}^{(c)} = \{t \in \Delta : f_t^i(c) = f_t^j(c)\}.$$

To prove the statement, it suffices to prove that $\Delta_{i,j}^{(c)}$ is a discrete set for any $c \in \mathcal{C}$ and $i \neq j$. Indeed, this implies that for Lebesgue almost all parameters $t \in \Delta$, we have that $\mathcal{C} \cap \mathrm{PPer}(f_t) = \emptyset$. Combining this with Theorem A, the statement follows.

Let us prove that $\Delta_{i,j}^{(c)}$ is a discrete set for any $c \in \mathcal{C}$ and $i \neq j$. For any $t \in \Delta$, f_t satisfies the condition (SC), which implies that no critical point of f_t is periodic. Hence, we only need to consider the case that $i > 0$ and $j > 0$.

Arguing by contradiction, assume that there exist positive integers $i < j$ and $c \in \mathcal{C}$ such that $\Delta_{i,j}^{(c)}$ is not a discrete set. Without loss of generality, we can assume that $0 \in \Delta_{i,j}^{(c)}$ is an accumulation point of $\Delta_{i,j}^{(c)}$. Recall that

$$\frac{\partial_t \xi_n^{(c)}(t)}{Df_t^n(f_t(c))} = \sum_{k=0}^{n} \frac{\partial_t F(f_t^k(c), t)}{Df_t^k(f_t(c))} =: M_n^{(c)}(t). \tag{4.3}$$

Since f_0 satisfies the condition (NV_0), there exists $N \geq 1$ such that for any $l, p \geq N$, we have

$$\frac{1}{2} \leq \frac{|M_l^{(c)}(0)|}{|M_p^{(c)}(0)|} \leq 2, \tag{4.4}$$

Since 0 is an accumulation point of $\Delta_{i,j}^{(c)}$, there exists a sequence of parameters $\{t_n\}_{n=1}^{\infty} \subset \Delta_{i,j}^{(c)}$ such that $\lim_{n\to\infty} t_n = 0$, which implies that

$$\partial_t \xi_{i-1}^{(c)}(0) = \partial_t \xi_{j-1}^{(c)}(0).$$

A similar argument shows that for any positive integers l and p, we have

$$\partial_t \xi_{i+l(j-i)-1}^{(c)}(0) = \partial_t \xi_{j+p(j-i)-1}^{(c)}(0).$$

Together with (4.3) and (4.4), we obtain that

$$1 \geq \frac{1}{2} \cdot \frac{|Df_0^{i+l(j-i)-1}(f_0(c))|}{|Df_0^{j+p(j-i)-1}(f_0(c))|}, \tag{4.5}$$

provided that l and p are large enough. Note that f_0 satisfies the condition (SC). Fix p large enough and let $l \to \infty$, we get a contradiction. The statement follows. □

4.1.3 Hyperbolic times

Recall that

$$\mathrm{A}(x, f, n) = \sum_{j=0}^{n-1} \frac{|Df^j(x)|}{\mathrm{dist}(f^j(x), \mathcal{C})}.$$

By Lemma 3.10, if

$$\frac{|Df^n(x)|}{\mathrm{A}(x, f, n)} \geq \eta,$$

then map f^n is a diffeomorphism with uniformly bounded distortion on a neighborhood of x which is mapped to a ball $B(f^n(x), \theta_0 \eta e^{-1})$, where θ_0 is a constant given by Lemma 3.10. Hence, we introduce the following definition.

Definition 4.1. Given $\eta > 0$, we say that n is a η-hyperbolic time for a point x under the iteration of f, if

$$\frac{|Df^n(x)|}{\mathrm{A}(x, f, n)} \geq \eta.$$

Proposition 4.2. *Assume that $f \in \mathcal{A}$ satisfies the conditions (CE) and (WR). Then the following hold.*

(i) There exist $\kappa \in (0,1)$ and $\eta > 0$ such that the following holds provided that n is large enough. For each $c \in \mathcal{C}$, there exists $\kappa n \leq m \leq n$ such that m is a η-hyperbolic time for $f(c)$ under the iteration of f.

(ii) Given $\tau > 0$ small and $\varepsilon > 0$ small, there exists $\kappa \in (0,1)$ and $\eta > 0$ such that the following holds provided that n is large enough. Let $x \in [0,1]$ be such that

$$\sum_{i=0}^{n-1} q_\varepsilon(f^i(x)) \leq \tau n,$$

then there exists $\kappa n \leq m \leq n$ such that m is a η-hyperbolic time for x under the iteration of f.

We shall need the following lemma which is [1, Lemma 3.1].

Lemma 4.3 (Pliss Lemma)**.** *Given $A_1 > A_2 > A_3$, let $\kappa = (A_2 - A_3)/(A_1 - A_3)$. Then, given any real numbers $a_1, \dots, a_n$ such that*

$$\sum_{k=1}^{n} a_k \geq A_2 n \text{ and } a_k \leq A_1 \text{ for any } 1 \leq k \leq n,$$

there exist $m > \kappa n$ and $1 < n_1 < n_2 < \cdots < n_m \leq n$ so that

$$\sum_{k=j+1}^{n_i} a_k \geq A_3(n_i - j) \text{ for every } 0 \leq j < n_i \text{ and } i = 1, \dots, m.$$

To prove Proposition 4.2, we prove the following result.

Lemma 4.4. *Consider $f \in \mathcal{A}$. For any $\lambda > 1$, there exists $\sigma > 0$ with the following property. For each $\delta > 0$, there exist $\eta > 0$ and $\kappa \in (0,1)$ such that for each $n \in \mathbb{N}$ and each $x \in [0,1]$, if*

$$|Df^n(x)| \geq \lambda^n$$

and

$$\sum_{i=0}^{n-1} \log |Df(f^i(x))| \cdot 1_{C(f,\delta)}(f^i(x)) \geq -\sigma n,$$

then there exists $\kappa n \leq m \leq n$ such that m is a η-hyperbolic time of x under the iteration of f.

Proof. Let $C_0 = \max_{[0,1]} |Df|$. Since $f \in \mathcal{A}$, there exists a constant $\rho > 0$ such that

$$\operatorname{dist}(x, \mathcal{C}) \geq \rho \cdot |Df(x)|^{(\ell_{min}-1)^{-1}}.$$

Given $\lambda > 1$, let

$$\sigma = \frac{(\log \lambda)^2 (\ell_{min} - 1)}{8 \log C_0}.$$

We shall prove the statement hold for these λ and σ.

To prove the statement, the strategy here is to use Lemma 4.3 twice, first for the sequence given by $a_i = \log |Df(f^{i-1}(x))|$, and then for the sequence given by $b_i = \log |Df(f^{i-1}(x))| \cdot 1_{C(f,\delta)}(f^{i-1}(x))$.

For the sequence a_i, we have

$$\sum_{i=1}^{n} a_i \geq (\log \lambda) \cdot n \text{ and } a_i \leq \log C_0 \text{ for every } 1 \leq i \leq n.$$

Let $A_1 = \log C_0$, $A_2 = \log \lambda$ and $A_3 = \frac{1}{2} \log \lambda$, applying Lemma 4.3, we obtain $\kappa_a = (\log \lambda)/(2 \log C_0 - \log \lambda)$.

For the sequence b_i, we have

$$\sum_{i=1}^{n} b_i \geq -\sigma \cdot n \text{ and } b_i \leq 0 \text{ for every } 1 \leq i \leq n.$$

Let $B_1 = 0$, $B_2 = -\sigma$ and $B_3 = -l \cdot \sigma$, applying Lemma 4.3 again, we obtain $\kappa_b = (l-1)/l > 0$, where $l = 2 \log C_0 / \log \lambda$.

Our conditions on l imply that $\kappa_a + \kappa_b > 1$. Let $\kappa = \kappa_a + \kappa_b - 1$. Then there exists $\kappa n \leq m \leq n$ such that

$$\sum_{i=j+1}^{m} a_i \geq A_3(m-j) \text{ and } \sum_{i=j+1}^{m} b_i \geq B_3(m-j)$$

hold for every $0 \leq j < m$. Hence, we obtain that

$$|Df^{m-j}(f^j(x))| \geq \lambda^{\frac{m-j}{2}} \text{ and } \operatorname{dist}(f^j(x), \mathcal{C})^{-1} \leq (\delta^{-1} + \rho^{-1} \cdot \lambda^{\frac{m-j}{4}}).$$

These imply that

$$\frac{\mathrm{A}(x, f, m)}{|Df^m(x)|} \leq \frac{1}{\delta(\lambda^{\frac{1}{2}} - 1)} + \frac{1}{\rho(\lambda^{\frac{1}{4}} - 1)} \leq \eta^{-1},$$

where

$$\eta = \frac{\delta\rho(\lambda^{\frac{1}{4}} - 1)}{\rho + \delta}.$$

Thus, the lemma is proved. □

Proof of Proposition 4.2. (i) Since f satisfies the condition (CE), there exists $\lambda > 1$ such that $|Df^n(f(c))| \geq \lambda^n$ holds for any $c \in \mathcal{C}$ and n large enough. Let $\sigma > 0$ be the constant given by Lemma 4.4 for these f and λ. Note that f satisfies the condition (WR). Hence, there exists $\delta > 0$ such that

$$\sum_{i=0}^{n-1} \log |Df(f^{i+1}(c))| \cdot 1_{C(f,\delta)}(f^{i+1}(c)) \geq -\sigma n$$

holds for any $c \in \mathcal{C}$ and n large enough. By Lemma 4.4, the statement (i) follows.

(ii) Let $\alpha > 0$ be as in Proposition 2.1 (iii), let $\lambda = e^{\alpha/2} > 1$ and let $\sigma > 0$ be the constant given by Lemma 4.4 for these f and λ. Assume $\tau > 0$ is a small constant such that

$$3\ell_{max} \cdot \tau \leq \alpha \text{ and } 2\ell_{max} \cdot \tau \leq \sigma.$$

Fix $\varepsilon > 0$ small such that the conclusion of Proposition 2.1 holds and $\Lambda(\varepsilon) \geq e^{\alpha}$, then there exists $\delta > 0$ such that $q_\varepsilon(y) \geq -\log(D_c(\varepsilon))$ holds for any $c \in \mathcal{C}$ and $y \in B(c, \delta)$.

Consider $x \in [0, 1]$ with

$$\sum_{i=0}^{n-1} q_\varepsilon(f^i(x)) \leq \tau n.$$

Let $0 \leq n_1 < n_2 < \cdots < n_m < n$ be all the integers such that $f^{n_j}(x) \in \widetilde{B}(\varepsilon)$. By Proposition 2.1 (iii) and (3.2), for any $0 < j < m$, we obtain that

$$|Df^{n_{j+1}-n_j}(f^{n_j+1}(x))| \geq \exp\Big(\alpha(n_{j+1} - n_j) - \ell_{max} q_\varepsilon(f^{n_{j+1}}(x))\Big).$$

Applying Proposition 2.1 (iii) again, we get that

$$|Df^{n_1}(x)| \geq A\varepsilon^{1-\ell_{max}^{-1}} \exp(\alpha n_1)$$

and

$$|Df^{n-n_m-1}(f^{n_m+1}(x))| \geq A\varepsilon^{1-\ell_{max}^{-1}} \exp(\alpha(n - n_m - 1)),$$

where A is as in Proposition 2.1 (iii). Hence, there exists $C > 0$ independent of n such that

$$|Df^n(x)| \geq C \exp((\alpha - \ell_{max}\tau)n) \geq \lambda^n,$$

provided that n is large enough. For any $c \in \mathcal{C}$ and $y \in B(c, \delta)$, by our conditions on δ, we obtain that

$$\log |Df(y)| > -\ell_{max} \cdot q_\varepsilon(y) + \log(D_c(\varepsilon)) > -2\ell_{max} \cdot q_\varepsilon(y).$$

This implies that

$$\sum_{i=0}^{n-1} \log |Df(f^i(x))| \cdot 1_{C(f,\delta)}(f^i(x)) \geq -\sigma n.$$

By Lemma 4.4, the statement (ii) follows. $\square$

4.1.4 Spectral decomposition

In this subsection, we will introduce some results for obtaining the relation between Theorem B and Theorem B*. These results were already known and we provide proofs of them for the reader's convenience.

Definition 4.2. An interval I is called *periodic of period* p, if $f^p(I) \subset I$ and the interiors of I and $f^i(I)$ are disjoint for each $1 \leq i < p$.

Definition 4.3. An open interval I is called *nice*, if $I \cap f^i(\partial I) = \emptyset$ for each $i \in \mathbb{N}$.

Then we have the following result.

Lemma 4.5. *If I is a periodic interval with period p, there exists an open nice interval J containing int(I) such that $f^i(I) \cap J = \emptyset$ holds for each $1 \leq i < p$.*

Proof. For each $p \in \mathbb{N}$ and each periodic interval I with period p, define

$$A = \bigcup_{n=0}^{\infty} f^{-n}(\text{int}(I)).$$

Let J be the component of A that contains $\text{int}(I)$. Since A is open, J is open.

We first prove that J is nice. Otherwise, there exist $x \in \partial J$ and $n \in \mathbb{N}$ such that $f^n(x) \in \text{int}(J) \subset A$. Then $x \in f^{-n}(A) \subset A$. Since A is open, it is in contradiction with the definition of J.

Let J' be the component of $\bigcup_{n=0}^{\infty} f^{-np}(\text{int}(I))$ that contains $\text{int}(I)$. We claim that $J' = J$. It is easy to get that $J' \subset J$. Assume $J' \neq J$, so there exist a positive integer $k \not\equiv 0 \pmod p$ and $n \in \mathbb{N}$ such that

$$f^{-np}(\text{int}(I)) \cap f^{-k}(\text{int}(I)) \neq \emptyset.$$

Combining this and $f^p(I) \subset I$, we obtain that there exists $1 \leq i < p$ such that

$$\mathrm{int}(I) \cap f^i(I) \neq \emptyset.$$

Since $\mathrm{int}(I)$ is an open interval and $f^i(I)$ is an interval, we get

$$\mathrm{int}(I) \cap \mathrm{int}(f^i(I)) \neq \emptyset.$$

This is a contradiction. Thus, the claim follows.

Similarly, we can prove that $f^i(I) \cap f^{-np}(\mathrm{int}(I)) = \emptyset$ holds for any $1 \leq i < p$ and $n \in \mathbb{N}$. Combining this, the definition of J' and the fact that $J' = J$, the statement follows. □

Definition 4.4. Let $J \subset T$ be open intervals such that $T \setminus J$ consists of intervals L and R. Define the cross-ratio of these intervals as

$$C(T, J) = \frac{|T||J|}{|L||R|}.$$

Let $J' \subset T'$ be components of $f^{-1}(J)$ and $f^{-1}(T)$. If f is a C^2 multimodal map with non-flat critical points, there exists $\gamma = \gamma(f) > 0$ such that

$$\frac{C(T, J)}{C(T', J')} \geq \begin{cases} 1 - \gamma|T| & \text{if } T' \text{ does not contain a critical point,} \\ \gamma & \text{if } T' \text{ contains a critical point,} \end{cases} \tag{4.6}$$

and

$$|Df(x)| \leq \frac{1}{\gamma} \frac{|f(T')|}{|T'|} \text{ for each } x \in T'; \tag{4.7}$$

see [34]. Suppose $(f_t)_{t \in [-1,1]}$ is a normalized regular family, then γ can be chosen uniformly in t.

Proposition 4.6. *Assume that f is a C^2 multimodal interval map with non-flat critical points and the critical points of f are not pre-periodic. There exists a constant $K > 1$ that only depends on $\gamma(f)$ and the number of critical points of f such that the following hold. If I is a periodic interval with period p, then for any $x \in \bigcup_{i=0}^{p-1} f^i(I)$, we have*

$$|Df^p(x)| \leq K.$$

In particular, if $(f_t)_{t \in [-1,1]}$ be a normalized regular family, then K can be chosen uniformly in t.

Proof. Let I be a periodic interval with period p. By Lemma 4.5, there exists an open nice interval J such that $\mathrm{int}(I) \subset J$ and $f^i(I) \cap J = \emptyset$ holds for any $1 \leq i < p$.

We claim that there exists a return domain I' to J with period p such that $\mathrm{int}(I) \subset \mathrm{int}(I')$. Note $f^p(I) \subset I$. Then the claim holds when $I \subset \mathrm{int}(J)$. We may assume $\mathrm{int}(I) = \mathrm{int}(J)$ and distinguish two cases. *Case 1.* $f^p(\mathrm{int}(I)) \subset \mathrm{int}(I)$, then the claim holds. *Case 2.* There exists $x \in \mathrm{int}(I)$ such that $f^p(x) \in \partial I$. Then the orbit of x intersect the critical points of f. Since $\mathrm{int}(I)$ is a nice interval and $f^p(I) \subset I$, then the boundary points of I are pre-periodic. It is an contradiction with the fact that the critical points of f are not pre-periodic. Thus, the claim follows.

Let I' be a return domain to J with period p such that $\mathrm{int}(I) \subset \mathrm{int}(I')$. By [34, Lemma 2], there exists an interval $M \supset J$ that is a ρ-scaled neighborhood of I' and that contains at most b of the intervals $f^i(I')$, $i = 1, \ldots, p-1$, where ρ and b are the constants only depend on $\gamma(f)$ and the number of critical points of f. Let $\{M_i\}_{i=0}^p$ be the collection of intervals such that $M_p = M$ and such that for $i \in \{0, 1, \ldots, p-1\}$, M_i are the pullbacks of M by f^{p-i} with $f^i(I') \subset M_i$. By [34, Lemma 3], there exists a constant κ which only depends on $\gamma(f)$ and the number of critical points of f such that the multiplicity of intersection of $\{M_i\}_{i=0}^p$ is bounded by κ. Let $\{I_i\}_{i=0}^p$ be a chain in the sense of [34] with $I_p = I$ and $I_0 \subset M_0$. Since $f^p(I) \subset I$, we have that $I \subset I_0$. Applying [34, Lemma 1], there exists a constant K which only depends on $\gamma(f)$ and the number of critical points such that $|Df^p(x)| \leq K$ holds for any $x \in I$. Note that for each i, $f^i(I)$ is periodic interval with period p. Thus, for any $x \in \bigcup_{i=0}^{p-1} f^i(I)$, we have that $|Df^p(x)| \leq K$.

Note that K is a constant which only depends on $\gamma(f)$ and the number of critical points of f. The statement follows. □

Remark. With a little modification, we can prove another version of Proposition 4.6: we can replace the condition that the critical points of f are not pre-periodic by the condition that $\mathrm{int}(I)$ is not nice. By the new version of Proposition 4.6, it is easy to conclude that f is not infinitely renormalizable, if $f \in \mathcal{A}$ satisfies the condition (SC).

Definition 4.5. A map $f \in C(I, I)$ is said to be *topologically mixing*, if for any non-empty open sets U and V, there exists an integer N such that for all $n > N$, we have $f^n(U) \cap V \neq \emptyset$. Furthermore, a map $f \in C(I, I)$ is said to be *topologically exact*, if for any non-empty open set U, there exists n such that $f^n(U) = I$.

Under some wild conditions, these two definitions are equivalent.

Lemma 4.7. *Consider a compact interval I and $f \in C^1(I, I)$ with all periodic points hyperbolic repelling. If f is topologically mixing, then f is topologically exact.*

Proof. Since $f : I \to I$ is topologically mixing, then f has a fix point $p \in \mathrm{int}(I)$. Note that all periodic points of f are hyperbolic repelling. These imply that there exists an open set $V \subset \mathrm{int}(I)$ such that $p \in V$ and such that $V \subset f(V)$.

To prove that statement, we only need to show that

$$\bigcup_{n=0}^{\infty} f^n(V) = I. \tag{4.8}$$

Indeed, by (4.8) and compactness of I, there exists N such that

$$I = \bigcup_{n=0}^{N} f^n(V) = f^N(V).$$

By topological mixing of f, for any non-empty open set $U \subset I$, there exists $k \in \mathbb{N}$ such that $V \subset f^k(U)$. Hence, we obtain that $f^{k+N}(U) = I$, which implies that f is topologically exact.

To this end, we shall prove (4.8). Denote $I = [a, b]$. By topological mixing of f, we can obtain that $(a, b) \subset \bigcup_{n=0}^{\infty} f^n(V)$. Arguing by contradiction, without loss of generality, we assume that $a \notin \bigcup_{n=0}^{\infty} f^n(V)$. Then $f^{-1}(a) \subset \{a, b\}$. We distinguish two cases.

Case 1. Assume $f(a) = a$. Then $f(b) \neq a$. Otherwise, $f^{-1}(b) \subset \mathrm{int}(I)$. Hence, there exists $c \in \mathrm{int}(I)$ such that $f(c) = b$, which implies that $f^2(c) = a$ and $a \in \bigcup_{n=0}^{\infty} f^n(V)$. Hence, we obtain that $f(a) = a$ and $f(b) \neq a$. Since all periodic points of f are hyperbolic repeller, we can obtain that $Df(a) > 1$. Combining this, $f^{-1}(a) \cap \mathrm{int}(I) = \emptyset$ and $f(b) \neq a$, there exists $\varepsilon > 0$ such that $f(x) > x$ holds for any $x \in (a, a + \varepsilon)$ and such that $f(x) > a + \varepsilon$ holds for any $x \in [a + \varepsilon, b]$. Let $W_1 = (a + \varepsilon/2, a + \varepsilon)$ and $W_2 = (a, a + \varepsilon/2)$, then we have that $f^n(W_1) \cap W_2 = \emptyset$ holds for any $n \in \mathbb{N}$. It is contradiction of topological mixing of f.

Case 2. Assume $f(b) = a$. In this case, we get $f^{-1}(b) \subset \{a, b\}$. Hence, $f(a) = b$. Since $|Df(a)| \cdot |Df(b)| = |Df^2(a)| > 1$, without loss of generality, we can assume that $|Df(a)| > 1$. Note that $f^{-1}(b) \cap \mathrm{int}(I) = \emptyset$ and $f(b) \neq b$. Similarly as in Case 1, we can get a contradiction. $\square$

Lemma 4.8. *Consider a positive integer r, a compact interval J and $f \in C^1(J, J)$ with all periodic points hyperbolic repelling. If $f^r : J \to J$ is transitive, there exist a periodic interval I with period p such that $f^p : I \to I$ is topologically exact and such that*

$$\bigcup_{n=0}^{r-1} f^n(J) = \bigcup_{n=0}^{p-1} f^n(I).$$

Proof. Let M be the component of $A := \bigcup_{n=0}^{\infty} f^n(J)$ that contains J and q be the minimal positive integer such that the intersection of interiors of $f^q(M)$ and M is non-empty. Then M is a periodic interval with period q.

Indeed, $f^r(J) \subset J$ and $J \subset M$, which implies that the intersection of interiors of $f^r(M)$ and M are non-empty. Hence, q is well defined. Since A is f-forward invariant, we have that $f^q(M) \subset A$. By the definition of M and q, we can conclude that $f^q(M) \subset M$. Moreover, we can get M is a periodic interval with period q.

Note that

$$\bigcup_{n=0}^{q-1} f^n(M) \subset \bigcup_{n=0}^{\infty} f^n(J) \subset \bigcup_{n=0}^{r-1} f^n(J) \subset \bigcup_{n=0}^{\infty} f^n(M) \subset \bigcup_{n=0}^{q-1} f^n(M),$$

which implies $A = \bigcup_{n=0}^{q-1} f^n(M)$. Combining this and $f^r : J \to J$ is transitive, we can obtain that $f^q : M \to M$ is transitive.

Applying [9, Lemma 8.3], there exist an interval I and a positive integer p such that the following hold:

- I is a periodic interval with period p;
- $f^p : I \to I$ is topologically mixing;
- $$\bigcup_{n=0}^{p-1} f^n(I) = A.$$

By Lemma 4.7, the statement follows. □

Proposition 4.9. *For any $f \in \mathcal{A}$ which satisfies the condition (SC), there exist a finite family of closed intervals $\{I_i\}_{i=1}^m$ and a finite family of positive integers $\{p_i\}_{i=1}^m$ with the following properties:*

(1) for each i, I_i is a periodic interval with period p_i and $f^{p_i} : I_i \to I_i$ is topologically exact;

(2) for each i, at least one of the intervals $I_i, \ldots, f^{p_i-1}(I_i)$ contains a critical point;

(3) for each $i \neq j$ the intersection of $\bigcup_{k=0}^{p_i-1} f^k(I_i)$ and $\bigcup_{k=0}^{p_j-1} f^k(I_j)$ is at most finite;

(4) for Lebesgue almost every $x \in [0,1]$, there exists i such that

$$\omega(x) = \bigcup_{k=0}^{p_i-1} f^k(I_i). \tag{4.9}$$

In particular, if $m = 1$, then $f : [0,1] \to [0,1]$ is ergodic.

Proof. Since f is of class C^3 with non-flat critical points, then f has no wandering interval. Combining this with the fact that all periodic orbits of f are hyperbolic repelling, we obtain that f has no homterval. By [24, Theorem 1], there is a decomposition $[0,1] = \bigcup_{i=1}^m E_i$ (mod 0) of $[0,1]$ into the finite union of invariant sets of positive Lebesgue measure such that $f|_{E_i}$ are ergodic (where (mod 0) here means that we ignore sets of measure zero). For each i, by [24, Theorem 2], there exists an attractor $\{A_i\}_i$ with the following properties:

- the limit set $\omega(x) = A_i$ holds for a.e. $x \in E_i$;
- each A_i contains a critical point;
- for any $i \neq j$, the intersection of A_i and A_j is at most finite.

Since f is not infinitely renormalizable, then A_i can not an attracting Cantor set. Moreover, for each i, there exists an acip μ_i supported on E_i. Note that $\mathrm{supp}(\mu_i)$ is a f-forward invariant set, which implies that $\mathrm{supp}(E_i) \subset A_i$. Hence, A_i can not be an absorbing Cantor attractor. Thus, by [24, Theorem 5], A_i is a cycle of transitive intervals. Applying Lemma 4.8, the statement follows. □

Corollary 4.10. *If $f \in \mathcal{A}$ satisfies the condition (SC), then $\overline{PPer(f)} = [0,1]$.*

Proof. Consider a compact interval I and a topological exact map $g \in C(I, I)$. We first prove that $\overline{PPer(g)} = I$. Indeed, for any open interval $U \subset I$, there exists $n \in \mathbb{N}$ such that $f^n(U) = I$. Then there exists $p \in \overline{U}$ such that $f^n(p) = p$. Hence, $\overline{PPer(g)} = I$.

Fix $f \in \mathcal{A}$ which satisfies the condition (SC). Let $\{I_i\}_i$ and $\{p_i\}_i$ be as in Proposition 4.9. By the above discussion, we have that $I_i \subset \overline{\mathrm{PPer}(f^{p_i}|_{I_i})} \subset \overline{\mathrm{PPer}(f)}$ holds for each i. Note that $\overline{\mathrm{PPer}(f)}$ is a f-invariant closed set. Combining this and (4.9), the statement follows. □

4.1.5 Uniqueness of acip and (LD) property

The aim of this subsection is to prove the following result.

Proposition 4.11. *Consider a normalized regular family $(f_t)_{t\in[-1,1]}$. Let $\Delta_{uL} \subset \Delta_0$ be the set of parameters t for which f_t admits a unique acip μ_t, μ_t is ergodic and f_t satisfies the property (LD). Assume t_0 is the parameter with the following properties:*

- *t_0 is a Lebesgue density point of Δ_0;*
- *f_{t_0} admits a unique acip;*
- *$f_{t_0} : [0,1] \to [0,1]$ is topologically exact.*

Then t_0 is a Lebesgue density point of the set Δ_{uL}.

In the following of this subsection, we fix a normalized regular family $(f_t)_{t\in[-1,1]}$. Without loss of generality, we assume that $t_0 = 0$. We divide the proof into two parts.

Uniqueness of acip

In the first part, we focus on the parameters t for which f_t has a unique acip μ_t and μ_t is ergodic. Note that the condition $f : [0,1] \to [0,1]$ is topologically exact plays no role in this part.

Lemma 4.12. *Consider $g \in \mathcal{A}$ which satisfies the condition (SC). There exist constants $C > 0$ and $\kappa > 0$ such that*

$$|g^{-n}(E)| \le C \cdot |E|^{\kappa} \tag{4.10}$$

holds for each Lebesgue measurable set E and each $n \in \mathbb{N}$. If A is a g-forward invariant set with positive Lebesgue measure, there exists an acip μ such that

$$|\mu(E)| \le \frac{C}{|A|}|E|^{\kappa} \tag{4.11}$$

holds for any Lebesgue measurable set E and such that supp$(\mu) \subset A$, where supp(μ) denote the support of μ. Moreover, if g admits a unique acip μ, then μ is ergodic.

Proof. Combining [11, Theorem 1] and [11, Theorem 2], there exist constants $C > 0$ and $\kappa > 0$ such that (4.10) holds for each Lebesgue measurable set E and each $n \in \mathbb{N}$.

Let A be a g-forward invariant set with positive Lebesgue measure and denote $g_A = g|_A$. By (4.10), for each $n \geq 1$, the measure

$$\nu_n = \frac{1}{n}\sum_{i=0}^{n-1}(g_A^i)_* \,\mathrm{Leb}$$

is absolutely continuous with respect to Lebesgue measure. Hence, any accumulation point ν of the set $\{\nu_n\}_n$ in the weak star topology is an absolutely continuous invariant measure. Let $\mu := \nu/|A|$. Then μ is an acip and (4.11) holds. Furthermore, we have $\mathrm{supp}(\mu) \subset A$.

Suppose g admits a unique acip μ, but μ is not ergodic. Then there exists an invariant subset A of $[0, 1]$ such that $0 < \mu(A) < 1$. From above, we can construct an acip ν with $\mathrm{supp}(\nu) \subset A$, which implies that $\nu(A) = 1$. Hence, μ and ν are two distinct acips. It is a contradiction. □

Lemma 4.13. *Consider $g \in \mathcal{A}$ which satisfies the condition (SC). Let $\{I_i\}_{i=1}^m$ and $\{p_i\}_{i=1}^m$ be given by Proposition 4.9 for g. Then $m = 1$ if and only if g has a unique acip μ.*

Proof. Assume g has a unique acip. For each i, let $A_i = \bigcup_{j=0}^{p_i-1} g^j(I_i)$, then A_i is a g-forward invariant set with positive Lebesgue measure. By Lemma 4.12 and Proposition 4.9 (3), we can obtain that $m = 1$.

Assume $m = 1$. Arguing by contradiction, we assume that g admits two acips μ and ν. By the ergodicity of g, we get μ and ν are ergodic. Hence, we obtain that $|\mathrm{supp}(\mu) \cap \mathrm{supp}(v)| = 0$. Note that the support of acip is a g-forward invariant closed set with positive Lebesgue measure. Combining this with Proposition 4.9 (4), we have $A_1 \subset \mathrm{supp}(\mu) \cap \mathrm{supp}(\nu)$. This is a contradiction. □

For $C > 0$, $\varepsilon > 0$ and $\varepsilon_0 \geq \varepsilon$, let $\mathcal{D}(C, \varepsilon, \varepsilon_0)$ be the collection of all triples (x, t, p)

with the following properties: $|t| \leq \varepsilon$, $\mathrm{dist}(x, \mathrm{CV}) \leq 4\varepsilon$, $f_t^{p-1}(x) \in \widetilde{B}(\varepsilon_0)$ and

$$\sum_{j=0}^{p-1} q_\varepsilon(f_t^j(x)) \leq C \cdot \#\{0 \leq j < p : f_t^j(x) \in \widetilde{B}(\varepsilon)\}.$$

Lemma 4.14. *Given $K > 1$ and $C > 0$, there exists $\varepsilon_0 > 0$ such that $|Df_t^p(x)| > K$ holds for any $\varepsilon \leq \varepsilon_0$ and $(x, t, p) \in \mathcal{D}(C, \varepsilon, \varepsilon_0)$.*

Proof. Given $K > 1$ and $C > 0$, by Proposition 2.1 (i), there exists $\varepsilon_0 > 0$ such that $|Df_t^p(x)| > K$ holds for each $\varepsilon \leq \varepsilon_0$ and each $(x, t, p) \in \mathcal{D}(C, \varepsilon, \varepsilon)$. In particular, $|Df_t^p(x)| > K$ holds for any $(x, t, p) \in \mathcal{D}(C, \varepsilon_0, \varepsilon_0)$.

For any $\varepsilon \leq \varepsilon_0$, define

$$K(\varepsilon) = \min \left\{ |Df_t^p(x)| : (x, t, p) \in \mathcal{D}(C, \varepsilon, \varepsilon_0) \right\}.$$

To complete the proof, it suffices to prove that $K(\varepsilon) \geq K$ implies that $K(\varepsilon') \geq K$ holds for any $\varepsilon' \in [\varepsilon/2, \varepsilon]$.

To this end, consider $\varepsilon' \in [\varepsilon/2, \varepsilon]$ and $(x, t, p) \in \mathcal{D}(C, \varepsilon', \varepsilon_0)$. Let $m \leq p$ be the maximal positive integer such that $f_t^{m-1}(x) \in \widetilde{B}(\varepsilon')$. Hence, $(x, t, m) \in \mathcal{D}(C, \varepsilon', \varepsilon')$, which implies that $|Df_t^m(x)| > K$. If $m = p$, we have finished the proof. Otherwise, by our conditions on m and ε', we obtain that the triple $(f_t^m(x), t, p - m) \in \mathcal{D}(C, \varepsilon, \varepsilon_0)$, which implies that $|Df_t^{p-m}(f^m(x))| \geq K(\varepsilon) > K$. Thus,

$$|Df_t^p(x)| = |Df_t^m(x)| \cdot |Df_t^{p-m}(f^m(x))| > K.$$

The statement follows. $\square$

Lemma 4.15. *Let K be the constant given by Proposition 4.6 for the normalized regular family $(f_t)_{t \in [-1,1]}$. Given $C > 0$, there exists $\varepsilon_0 > 0$ such that the following holds for any $\varepsilon < \varepsilon_0$ and $t \in X_\varepsilon(C) \cap \Delta_0$. Let I be a periodic interval with period p for f_t such that $\bigcup_{j=0}^{p-1} f_t^j(I) \cap \mathcal{C} \neq \emptyset$. Denote $\rho = \max_{j=0}^{p-1} |f_t^j(I)|$, then $\rho \geq \varepsilon_0$.*

Proof. Fix $C > 0$, let ε_0 be the constant given by Lemma 4.14 for these K and C. We shall prove the statement holds for this ε_0. Arguing by contradiction, assume that ε, t and I are as in this lemma, but $\rho < \varepsilon_0$. For any $c \in \mathcal{C} \cap \bigcup_{j=0}^{p-1} f_t^j(I)$, by our conditions on ε, t and I, we can obtain that the triple $(f_t(c), t, p) \in \mathcal{D}(C, \varepsilon, \varepsilon_0)$, which implies that $|Df_t^p(f_t(c))| > K$. This is a contradiction of the choice of K. $\square$

Lemma 4.16. *Let $\Delta_u \subset \Delta_0$ be the collection of the parameters t for which f_t admits a unique acip μ_t and μ_t is ergodic. Then 0 is a Lebesgue density point of the set Δ_u.*

Proof. Let $K > 1$ be the constant given by Proposition 4.6 for the regular family $(f_t)_{t\in[-1,1]}$. Arguing by contradiction, assume 0 is not a Lebesgue density point of the set Δ_u. Then there exists $\delta > 0$ such that

$$\limsup_{\varepsilon\to 0} \frac{|[-\varepsilon,\varepsilon] \cap (\Delta_0 \setminus \Delta_u)|}{2\varepsilon} > \delta.$$

Note that $f \in \mathcal{A}$ satisfies the condition (SC). By the Reduced Theorem A, let $C > 0$ be the constant such that $|[-\varepsilon,\varepsilon] \cap X_\varepsilon(C)| \geq (1-\delta)2\varepsilon$ holds for $\varepsilon > 0$ small enough, where $X_\varepsilon(C)$ is as defined in (3.7). Let $\varepsilon_0 = \varepsilon_0(K, C)$ be given by Lemma 4.15. Hence, there exist sequences of positive real numbers $\{\varepsilon_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ with the following properties: $\varepsilon_n \leq \varepsilon_0$, $\lim_n \varepsilon_n = \lim_n t_n = 0$ and $t_n \in X_{\varepsilon_n}(C) \cap (\Delta_0 \setminus \Delta_u)$.

Since $t_n \in \Delta_0 \setminus \Delta_u$, by Lemmas 4.13 and 4.12, let I_n and J_n be given by Proposition 4.9 for f_{t_n} with periods s_n and p_n respectively. Replacing $\{I_n\}_n$ and $\{J_n\}_n$ by their subsequences, if necessary, we can assume that the limits $\lim_n I_n$ and $\lim_n J_n$ exist. Denote $I = \lim_n I_n$ and $J = \lim_n J_n$. By Lemma 4.15, I and J are the intervals with positive Lebesgue measure. Furthermore, we can obtain that the interior of $f^i(I)$ and $f^j(J)$ are disjoint for any i and j. Hence, there exist acips μ and ν which are supported on the sets $\bigcup_{i=0}^\infty f^i(I)$ and $\bigcup_{j=0}^\infty f^j(J)$ respectively. This is a contradiction of the uniqueness of acip. The statement follows. $\square$

The (LD) property

The following lemma was proved in [16], in the unimodal case. We provide a proof for reader's convenience.

Lemma 4.17. *Consider $g \in \mathcal{A}$ which satisfies the condition (CE). Assume g admits a unique acip μ and let I be the periodic interval given by Lemma 4.13 with period p. For each $\phi \in C^2([0,1],\mathbb{R})$ and each $\delta > 0$, there exists $C > 0$ and $\rho > 0$ such that*

$$\left|\left\{x \in \bigcup_{j=0}^{p-1} g^j(I) : \left|\frac{1}{n}\sum_{i=0}^{n-1} \phi(g^i(x)) - \int \phi d\mu\right| > \delta\right\}\right| < Ce^{-\rho n}. \tag{4.12}$$

Proof. As the same process in [25], to prove the statement, we only need to consider the case that $p = 1$. In this case, $g : I \to I$ is topologically mixing.

Given a sufficiently small nice couple $(\widehat{V}, V)$ of $g|_I$, we say that an integer $m \geq 1$ is a good time for a point x, if $g^m(x) \in V$ and if the pullback of $\widehat{V}$ containing x is diffeomorphic. Let D denote the set of all those points in V having a good time. For each $x \in D$, we denote by $m(x)$ the least good time of x. Then we can define the canonical induced map G associated to the nice couple $(\widehat{V}, V)$ by $G(x) = g^{m(x)}(x)$, see [33]. Since g satisfies the exponential shrinking condition, by [28, Thereom E], there exists $\lambda > 1$ such that $|DG(x)| \geq \lambda^{m(x)}$. We denote by $J(G)$ the maximal invariant set of G, which is equal to the set of all those points in V having infinitely many good times. By [33, Lemma 6.2], we obtain that $|V \setminus J(G)| = 0$.

For each $c \in \mathcal{C}(g|_I)$, let V^c and $\widehat{V}^c$ be the connected component of V respectively $\widehat{V}$ containing c. Let $\widehat{G}$ be the first return map of G to V^c. By topological exact, for Lebesgue almost all $x \in J(G)$, there exists $R \geq 1$ such that $\widehat{G}(x) = g^R(x)$ with $|Dg^R(x)| \geq \lambda^R$. Let J_x be the pullback of V^c along the orbit of x. Then $g^R : J_x \to V^c$ is a diffeomorphism and its distortion bounded by the constant which only depends on V and $\widehat{V}$. Hence, we constructed an at most countable partition $\{J_j\}_j$ of a full Lebesgue measure subset of V^c into intervals, such that R_j is a constant on J_j. Then $\widetilde{G} : \bigcup_j J_j \to V^c$ which defined by $\widetilde{G} = g^{R_j} : J_j \to V^c$ is a Young tower with exponential tail. By [25, Theoreom 2.1], for each $\phi \in C^2([0,1], \mathbb{R})$ and each $\delta > 0$, there exists a constant $C_1 > 0$ and $\rho > 0$ such that

$$\mu\left(\left\{x : \left|\frac{1}{n}\sum_{i=0}^{n-1} \phi(g^i(x)) - \int \phi d\mu\right| > \delta\right\}\right) < C_1 e^{-\rho n}.$$

Note $\mathrm{supp}(\mu) = I$. Hence, to prove the statement, we only need to show that $\frac{d\mu}{d\,\mathrm{Leb}}$ is uniformly bounded below on its support.

Let ν be an acip of $\widetilde{G}$. Then

$$\mu = \sum_{j=1}^{\infty} \sum_{i=0}^{R_j - 1} g_*^i \nu_j,$$

where $\nu_j(A) = \nu(A \cap J_j)$. By our construction, we have that $\frac{d\nu}{d\,\mathrm{Leb}}$ is uniformly bounded below on $\bigcup_j J_j$. This implies that $\frac{d\mu}{d\,\mathrm{Leb}}$ is uniformly bounded below on J_1. Since $g : I \to I$ is topologically exact, there exists $r \in \mathbb{N}$ such that $g^r(J_1) = I$. Hence, $\frac{d\mu}{d\,\mathrm{Leb}}$ is uniformly bounded below on I. The statement follows. $\square$

Lemma 4.18. *Consider $g \in \mathcal{A}$ which satisfies the condition (CE) and has a unique* *acip μ. Let I be the periodic interval given by Lemma 4.13 with period p. If for any $c \in \mathcal{C}(g)$, there exists $r \in \mathbb{N}$ such that $g^r(c) \in int(I)$, then g satisfies the property (LD).*

Proof. Denote $A = \bigcup_{j=0}^{p-1} g^j(I)$. Fix $\phi \in C^2([0,1],\mathbb{R})$. For $x \in [0,1]$, $n \in \mathbb{N}$ and $\delta > 0$, we define

$$B(n,x) = \left| \frac{1}{n} \sum_{i=0}^{n-1} \phi(g^i(x)) - \int \phi d\mu \right|$$

and

$$E_n(\delta) = \{x \in [0,1] : B(n,x) > \delta\}.$$

By Lemma 4.17, for each $\delta > 0$, there exist $C_1 > 0$ and $\rho_1 > 0$ such that

$$|E_n(\delta/2) \cap A| \le C_1 e^{-\rho_1 n}. \tag{4.13}$$

For any n large enough, fix m such that $(8C_0)^{-1}\delta n < m < (4C_0)^{-1}\delta n$, where $C_0 = \max_x |\phi(x)|$. Note that $B(m,x) \le 2C_0$ and

$$B(n,x) \le \left(\frac{m}{n}\right) \cdot B(m,x) + \left(\frac{n-m}{n}\right) \cdot B(n-m, g^m(x)),$$

which implies that

$$E_n(\delta) \cap \bigcup_{i=0}^{m-1} g^{-i}(A) \subset g^{-m}\left(E_{n-m}(\delta/2) \cap A\right).$$

By (4.13) and (4.10), we obtain that

$$\left| E_n(\delta) \cap \bigcup_{i=0}^{m-1} g^{-i}(A) \right| \le CC_1^{\kappa} \cdot e^{-\rho_1 \kappa (n-m)}. \tag{4.14}$$

By our conditions, there exists $R \in \mathbb{N}$, putting

$$U = \bigcup_{i=0}^{R} g^{-i}(\mathrm{int}(I)),$$

such that $\mathcal{C}(f) \subset U$. Hence,

$$[0,1] \setminus \bigcup_{i=0}^{m} g^{-i}(A) \subset [0,1] \setminus \bigcup_{i=0}^{m-R} g^{-i}(U).$$

Due to Máñe Theorem, there exist $C_2 > 0$ and $\rho_2 > 0$ such that

$$|\{x \in [0,1] : g^j(x) \notin U \text{ for } 0 \leq j < n\}| \leq C_2 e^{-\rho_2 n}. \tag{4.15}$$

Note that

$$E_n(\delta) \subset \left([0,1] \setminus \bigcup_{i=0}^{m} g^{-i}(A)\right) \cup \left(E_n(\delta) \cap \bigcup_{i=0}^{m} g^{-i}(A)\right) \cup \Big(E_n(\delta) \cap A\Big).$$

Together with (4.13), (4.14) and (4.15), the statement follows. □

Proof of Proposition 4.11. Assume $t \in \Delta_u \setminus \Delta_{uL}$. Let I be the periodic interval given by Lemma 4.13 for f_t with period p. We first prove that there exists $c \in \mathcal{C}$ such that $c \notin \bigcup_{k=0}^{\infty} f_t^k(I)$. Otherwise, for any $c \in \mathcal{C}$, there exists $k \in \mathbb{N}$ such that $c \in f_t^k(I)$. Due to $f_t^p(I) \subset I$, there exists $r_0 \in \mathbb{N}$ such that $f_t^{r_0+lp}(c) \in I$ holds for any $l \in \mathbb{N}$. Note that $\mathrm{PPer}(f_t) \cap \mathcal{C} = \emptyset$. Hence, there exists $r \in \mathbb{N}$ such that $f_t^r(c) \in \mathrm{int}(I)$. By Lemma 4.18, f_t satisfies the property (LD). Contradiction.

By Lemma 4.16, 0 is a Lebesgue density point of the set Δ_u. The rest of the proof is similar as the one in Lemma 4.16. In order to modify the proof of Lemma 4.16 in the text to that of Proposition 4.11, we have to replace the set $\Delta_0 \setminus \Delta_u$ by the set $\Delta_u \setminus \Delta_{uL}$. Then there exists $\{t_n\}_{n=1}^{\infty} \subset \Delta_u \setminus \Delta_{uL}$, such that the following hold: $\lim_n t_n = 0$ and $I = \lim_n I_n$ is an interval of positive Lebesgue measure, where I_n is the periodic interval given by Lemma 4.13 for f_{t_n}.

By the discussion above, replacing $\{t_n\}_n$ and $\{I_n\}_n$ by their subsequences, if necessary, we can assume that there exists $c \in \mathcal{C}$ such that $c \notin \bigcup_{k=0}^{\infty} f_{t_n}^k(I_n)$ holds for any n. This implies that $c \notin \bigcup_{k=0}^{\infty} \mathrm{int}(f^k(I))$. Since $f : [0,1] \to [0,1]$ is topologically exact, then $c \in \{0,1\}$. This is a contradiction of $\mathcal{C} \subset (0,1)$. Thus, the statement follows. □

4.1.6 Proof of Theorem B

In this subsection, we will devote to the proof of Theorem B. We first define some new parameter sets. Combining Proposition 4.2 and Proposition 3.14, for any $t \in \Delta_0$, there exists $\kappa \in (0,1)$ and $\eta > 0$ with the following property. For any $c \in \mathcal{C}$ and any N large enough, there exist $n > \kappa \cdot N$ and e-bounded c-parameter box B_n of order n centered at point t with $|B_n| \to 0$ as $n \to \infty$, such that

$$|\xi_n^{(c)}(B_n)| \geq \eta. \tag{4.16}$$

For any $\eta > 0$, let $\Delta_{0,\eta} \subset \Delta_0$ be the collection of parameters t for which there exists $\kappa \in (0,1)$ such that (4.16) holds. Furthermore, for any $c \in \mathcal{C}$, let $\Delta_*^{(c)} \subset \Delta_0$ be the collection of parameters t for which the asymptotic distribution of c exists and equals one of the ergodic acips.

To prove Theorem B, it suffices to prove that for any $\eta > 0$ and any Lebesgue density point t_* of the set $\Delta_{0,\eta}$, t_* is not a Lebesgue density point of the set $\Delta_{0,\eta} \backslash \Delta_*^{(c)}$. In the following, we will fix a normalized regular family $f_t : [0,1] \to [0,1], t \in [-1,1]$ and $\eta > 0$. Denote $F(x,t) = f_t(x)$. Without loss of generality, we assume that $t_* = 0$ is a Lebesgue density point of $\Delta_{0,\eta}$.

Lemma 4.19. *Let $I = [a,b]$ be a periodic interval of period p such that $f^p : I \to I$ is topologically exact. Then there exists $\varepsilon_0 > 0$, $a(t) \in C^2([-\varepsilon_0, \varepsilon_0], [0,1])$ and $b(t) \in C^2([-\varepsilon_0, \varepsilon_0], [0,1])$ with the following properties:*

- *$a(0) = a$ and $b(0) = b$;*
- *for any $|t| \leq \varepsilon_0$, $f_t^p(I(t)) \subset I(t)$;*
- *for any $|t| \leq \varepsilon_0$, $\mathcal{C}(f_t^p|_{I_t}) \cap \partial I_t = \emptyset$ and $\#\mathcal{C}(f^p|_I) = \#\mathcal{C}(f_t^p|_{I(t)})$,*

where $I(t) = [a(t), b(t)]$.

Proof. Denote $g = f^p|_I$. Since $g : I \to I$ is topologically exact, for any non-empty open subset U of $\mathrm{int}(I)$, there exists $n \in \mathbb{N}$ such that $g^n(U) = I$. Hence, there exists $x \in U$ such that $g^n(x) = a$. Let $m \leq n$ be the minimal integer such that $g^m(x) \in \partial(I)$, then $c_1 = g^{m-1}(x) \in \mathcal{C}(g)$. Without loss of generality, we assume that $g^m(x) = a$. We distinguish two cases.

Case 1. Assume $g(a) \in \partial I$. Note that $\mathcal{C} \cap \mathrm{PPer}(f) = \emptyset$, which implies that $\mathcal{C}(g) \cap \mathrm{PPer}(g) = \emptyset$. Then $g(a) = b$ and $g(b) \in \mathrm{int}(I)$. In this case, we have

$\mathcal{C}(g) \cap \partial I = \emptyset$. We define $a(t) = \min\{f_t^p(c) : c \in g^{-1}(a)\}$ and

$$b(t) = \max\left\{ \max\{f_t^p(c) : c \in g^{-1}(b)\}, \max\{f_t^{2p}(c) : c \in g^{-1}(a)\}\right\}.$$

We can easy to check that $f_t^p(I(t)) \subset I(t)$ holds for $|t|$ small enough.

Case 2. Assume $g(a) \in \mathrm{int}(I)$. Let U and n are defined as above. Then there exists $y \in U$ such that $g^n(y) = b$. Note that $g(a) \neq b$. Hence, there exists $c_2 \in \mathcal{C}(g)$ such that $g(c_2) = b$. In this case, $\partial I \subset g(\mathcal{C}(g))$. Hence, $\mathcal{C}(g) \cap \partial I = \emptyset$ and $g(b) \neq b$. If $g(b) \in \mathrm{int}(I)$, we define $a(t) = \min\{f_t^p(c) : c \in g^{-1}(a)\}$ and $b(t) = \max\{f_t^p(c) : c \in g^{-1}(b)\}$. If $g(b) = a$, this is the case 1 actually.

Note that $\mathcal{C}(g) \cap \partial I = \emptyset$ in both cases. Then let $|t|$ be small enough, the statement follows. □

Let $\{I_i\}_{i=1}^m$ and $\{p_i\}_{i=1}^m$ be given by Proposition 4.9 for f. By Lemma 4.19, we can define $I_i(t)$ for each i and $|t|$ small.

Lemma 4.20. *There exists $\varepsilon_0 > 0$ such that the following holds for any $c \in \mathcal{C}$. For Lebesgue almost all $t \in \Delta_{0,\eta} \cap [-\varepsilon_0, \varepsilon_0]$, there exists $r \in \mathbb{N}$ and $1 \le i \le m$ such that $f_t^r(c) \in int(I_i(t))$.*

Proof. Fix $c \in \mathcal{C}$. To prove the statement, we only need to show that for Lebesgue almost all $t \in \Delta_{0,\eta} \cap [-\varepsilon_0, \varepsilon_0]$, there exist $r \in \mathbb{N}$ and $1 \le i \le m$ such that $f_t^r(c) \in I_i(t)$. Indeed, if $f_t^r(c) \in I_i(t)$, then $f_t^{r+p_ik}(c) \in I_i(t)$ holds for any $k \in \mathbb{N}$. Combining this with the fact that $\mathcal{C} \cap \mathrm{PPer}(f_t) = \emptyset$, the statement follows.

By Proposition 4.9, there exists $R \in \mathbb{N}$ such that

$$\left|[0,1] \setminus \left(\bigcup_{i=1}^{m} \bigcup_{k=-R}^{p_i-1} f^k(I_i)\right)\right| < \frac{\eta}{8}.$$

Let $I_i = [a_i, b_i]$. Fix $\delta > 0$ small such that $|I_i| > 5\delta$ and such that

$$\left|[0,1] \setminus \left(\bigcup_{i=1}^{m} \bigcup_{k=-R}^{p_i-1} f^k(J_i)\right)\right| < \frac{\eta}{4},$$

where $J_i = [a_i + \delta, b_i - \delta]$. Let $I_i(t) = [a_i(t), b_i(t)]$, $J_i(t) = [a_i(t) + \delta, b_i(t) - \delta]$ and $P = \max_{i=1}^m p_i$. By continuity, there exists $\varepsilon_0 > 0$ with the following properties:

- for each i and $|t| < \varepsilon_0$, $|a_i(t) - a_i| < \delta/4$ and $|b_i(t) - b_i| < \delta/4$;

- for any $r \le R+P$, $|t| < \varepsilon_0$ and $x \in [0,1]$, $|f_t^r(x) - f^r(x)| < \delta/4$;
- for each $|t| < \varepsilon_0$,
$$\left|[0,1] \setminus \left(\bigcup_{i=1}^{m} \bigcup_{k=-R}^{p_i-1} f_t^k(J_i(t))\right)\right| < \frac{\eta}{2}.$$

We shall prove the statement holds for this constant ε_0. Let

$$\Delta_{bad} \subset \Delta_{0,\eta} \cap [-\varepsilon_0, \varepsilon_0]$$

be the collection of parameters t for which $f_t^r(c) \notin \bigcup_{i=1}^m I_i(t)$ holds for any $r \in \mathbb{N}$. To prove the statement, it suffices to prove that $|\Delta_{bad}| = 0$. Arguing by contradiction, assume that $t_0 \in \Delta_{bad}$ is a Lebesgue density point of the set Δ_{bad}. By the definition of $\Delta_{0,\eta}$, for any N large enough, there exist $\kappa \in (0,1)$, $n > \kappa \cdot N$ and e-bounded c-parameter box B_n of order n centered at point t_0 such that $|\xi_n^{(c)}(B_n)| \ge \eta$. Then

$$\left|\xi_n^{(c)}(B_n) \cap \left(\bigcup_{i=1}^{m} \bigcup_{k=-R}^{p_i-1} f_t^k(J_i(t_0))\right)\right| > \frac{\eta}{2}.$$

If $t \in B_n$ with $\xi_n^{(c)}(t) \in \bigcup_{i=1}^m \bigcup_{k=-R}^{p_i-1} f_t^k(J_i(t_0))$, then there exist $1 \le i \le m$ and $r \le R+P$ such that $f_{t_0}^r \circ \xi_n^{(c)}(t) \in J_i(t_0)$. By our conditions on δ and ε_0, we can obtain that $f_t^r \circ \xi_n^{(c)}(t) \in I_i(t)$, which implies that $t \notin \Delta_{bad}$. By the bounded distortion property of $\xi_n^{(c)}|_{B_n}$, it follows

$$\frac{|\Delta_{bad} \cap B_n|}{|B_n|} \le 1 - \frac{1}{2e}.$$

This is a contradiction. $\square$

Then we can construct a collection of normalized regular families. Let $\varepsilon_0 > 0$ be the small constant such that the conclusions of Lemmas 4.19 and 4.20 hold. For each $1 \le i \le m$, we can find a C^2 family $h_{i,t}$, $t \in [-\varepsilon_0, \varepsilon_0]$, of diffeomorphisms from $I_i(t)$ onto $[0,1]$, such that $g_{i,t}(x) = h_{i,t} \circ f_t^{p_i} \circ h_{i,t}^{-1} : [0,1] \to [0,1]$, $t \in [-\varepsilon_0, \varepsilon_0]$ is a normalized regular family. We denote $G_i(x,t) = g_{i,t}(x)$. We shall use this collection of normalized regular families and Theorem B* to prove Theorem B.

Proof of Theorem B. We will use notation $\Delta_0(F)$ to denote parameter set Δ_0 associated the family F. Similarly, we can define $\Delta_*(F)$, $\Delta_{uL}(F)$, etc. As the discussion at the beginning of this subsection, we aim to prove that 0 is not a Lebesgue density point of the set $\Delta_{0,\eta}(F) \setminus \Delta_*^{(c)}(F)$, provided that 0 is a Lebesgue density point of the set $\Delta_{0,\eta}(F)$.

Let G_i be defined as above. To prove Theorem B, it suffices to show that

$$(\Delta_{0,\eta}(F) \setminus \Delta_*^{(c)}(F)) \cap [-\varepsilon, \varepsilon] \subset \bigcup_{i=1}^{m} \Big(\Delta_0(G_i) \setminus \Delta_{uL}(G_i)\Big) \cap [-\varepsilon, \varepsilon] \pmod 0 \tag{4.17}$$

holds for any $\varepsilon > 0$ small enough.

Indeed, for any $t \in \Delta_{0,\eta}(F) \cap [-\varepsilon_0, \varepsilon_0]$, $t \in \Delta_0(G_i)$ holds for each $1 \leq i \leq m$. Note that 0 is a Lebesgue density point of the set $\Delta_{0,\eta}(F)$. This implies that 0 is a Lebesgue density point of the set $\Delta_0(G_i)$ for each $1 \leq i \leq m$. Furthermore, we have that $g_{i,0} : [0,1] \to [0,1]$ is topologically exact and $g_{i,0}$ admits a unique acip. By Proposition 4.11, 0 is a Lebesgue density point of the set $\Delta_{uL}(G_i)$. Together with (4.17), the statement follows.

Let us prove (4.17). By Lemma 4.20, for Lebesgue almost all $t_0 \in \Delta_{0,\eta}(F) \cap [-\varepsilon_0, \varepsilon_0]$, there exist $r \in \mathbb{N}$ and $1 \leq i \leq m$ such that $f_{t_0}^r(c) \in \operatorname{int}(I_i(t_0))$, which implies that there exists $\delta > 0$ such that $f_t^r(c) \in I_i(t)$ holds for any $t \in [t_0 - \delta, t_0 + \delta]$. We consider the parameter family $g_{i,t}(x) : [0,1] \to [0,1]$, $t \in [t_0 - \delta, t_0 + \delta]$ and let $z(t) = h_{i,t} \circ f_t^r(c)$. By Propositions 3.14 and 4.2, we can get that for any $t \in \Delta_0(G_i) \cap [t_0 - \delta, t_0 + \delta]$, z satisfies the property $(*)$ at parameter t for G_i. This implies that

$$\Delta_{uL}(G_i) \cap [t_0 - \delta, t_0 + \delta] \subset \Lambda(G_i) \cap [t_0 - \delta, t_0 + \delta].$$

Applying Theorem B*, we can get

$$|\,(\Lambda(G_i) \setminus \Lambda_*(G_i)) \cap [t_0 - \delta, t_0 + \delta]| = 0.$$

Besides, if $t \in \Lambda_*(G_i) \cap [t_0 - \delta, t_0 + \delta]$, then the asymptotic distribution of $z(t)$ for $g_{i,t}$ equals μ_t, which is the unique ergodic acip for $g_{i,t}$. Let

$$\nu_t = \frac{1}{p_i} \sum_{j=0}^{p_i - 1} \left(f^j \circ h_{i,t}^{-1}\right)_* \mu_t.$$

Then ν_t is an ergodic acip for f_t. Furthermore, we have that the asymptotic distribution of c for f_t equals ν_t, which implies that $t \in \Delta_*^{(c)}(F)$. Thus, (4.17) follows. □

4.1.7 Proof of Theorem B*

In this subsection, we will introduce the Reduced Theorem B* and deduce Theorem B* from the Reduced Theorem B*. So let (f_t), z, Λ, Λ_* and μ_t be as in Theorem B*.

Given $\delta > 0$ and $\phi \in C^2([0,1], \mathbb{R})$, let

$$\Lambda_*(\phi, \delta) = \left\{ t \in \Lambda : \limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=0}^{n-1} \phi(f_t^i(z(t))) - \int \phi d\mu_t \right| < \delta \right\}, \tag{4.18}$$

and let

$$E_n(\phi, \delta) = \left\{ t \in \Lambda : \left| \frac{1}{n} \sum_{i=0}^{n-1} \phi(f_t^i(z(t))) - \int \phi d\mu_t \right| \geq \delta \right\}. \tag{4.19}$$

Then we have that

$$\Lambda \setminus \Lambda_*(\phi, \delta) = \limsup_{n\to\infty} E_n(\phi, \delta). \tag{4.20}$$

If $t \in \Lambda$, then by hypothesis, f_t satisfies the property (LD). Together with Lemma 4.12, we have the following estimates. Given $\phi \in C^2([0,1], \mathbb{R})$ and $\delta > 0$, there exist $C_t > 0$, $\kappa_t > 0$ and $\rho_t > 0$ such that

$$|\mu_t(A)| \leq C_t \cdot |A|^{\kappa_t} \tag{4.21}$$

and

$$\left| \left\{ x \in [0,1] : \left| \frac{1}{n} \sum_{i=0}^{n-1} \phi(f_t^i(x)) - \int \phi d\mu_t \right| > \frac{\delta}{4} \right\} \right| \leq C_t e^{-\rho_t n} \tag{4.22}$$

hold for each $n \in \mathbb{N}$ and each Lebesgue measurable set A. For any positive integer Θ, we define

$$\Lambda_\Theta(\phi, \delta) = \{t \in \Lambda : C_t \leq \Theta,\ \kappa_t \geq \Theta^{-1} \text{ and } \rho_t \geq \Theta^{-1}\}. \tag{4.23}$$

Note that $\bigcup_{\Theta=1}^{\infty} \Lambda_\Theta(\phi, \delta) = \Lambda$.

To introduce the Reduced Theorem B*, we shall need the following results:

Lemma 4.21. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers with $\sum_{n=1}^{\infty} a_n < \infty$. Let A and E_n be Lebesgue measurable subsets of $[-1,1]$. Suppose for any $x \in A$*

there exists $N_x \in \mathbb{N}$, *such that*

$$\inf_{r\in(0,1)} \frac{|B(x,r)\cap E_n|}{|B(x,r)|} < a_n$$

hold for any $n > N_x$. *Then*

$$\left|\limsup_{n\to\infty} E_n \cap A\right| = 0.$$

Proof. Denote $F_i = \{x \in A : N_x < i\}$. It suffices to show that for any $i \in \mathbb{N}$, we have

$$\left|\limsup_{n\to\infty} E_n \cap F_i\right| = 0. \tag{4.24}$$

Fix $i \in \mathbb{N}$, for each $n \geq i$ and $x \in F_i$, there exists $r_{n,x} > 0$ such that

$$\frac{|B(x,r_{n,x})\cap E_n|}{|B(x,r_{n,x})|} < a_n. \tag{4.25}$$

For any $n \geq i$, by Besicovic's covering lemma, there exists a subfamily of

$$\{B(x,r_{n,x}) : x \in F_i\}$$

with uniformly bounded intersection multiplicity forms a covering of F_i. Combining this and (4.25), there exists a constant $C > 0$ such that

$$m^*(E_n \cap F_i) \leq C a_n,$$

where m^* denotes the Lebesgue outer measure. Hence,

$$\lim_{k\to\infty} m^*\Big(\bigcup_{n=k}^{\infty} E_n \cap F_i\Big) = 0.$$

The statement follows. □

Reduced Theorem B*. Let $F = (f_t)_{t\in[-1,1]}$ be a normalized regular family of interval maps. Given $\phi \in C^2([0,1],\mathbb{R})$, $\delta > 0$ and positive integer Θ, for each $t_0 \in \Lambda_\Theta(\phi,\delta)$, we have

$$\inf_{r\in(0,1)} \frac{|B(t_0,r)\cap E_n(\phi,\delta)\cap\Lambda_\Theta(\phi,\delta)|}{2r} \leq \frac{1}{n^2},$$

holds for n large enough.

Proof of Theorem B.* Given $\phi \in C^2([0,1],\mathbb{R})$, $\delta > 0$ and a positive integer Θ, for any $t_0 \in \Lambda_\Theta(\phi,\delta)$, by Reduced Theorem B*, we obtain that

$$\inf_{r\in(0,1)} \frac{|B(t_0,r)\cap E_n(\phi,\delta)\cap\Lambda_\Theta(\phi,\delta)|}{2r} \le \frac{1}{n^2}$$

holds for n large enough. Applying Lemma 4.21 for $E_n = E_n(\phi,\delta)\cap\Lambda_\Theta(\phi,\delta)$ and (4.20) we obtain that

$$|\Lambda_\Theta(\phi,\delta)\setminus\Lambda_*(\phi,\delta)| = 0.$$

This implies that $|\Lambda\backslash\Lambda_*(\phi,\delta)| = 0$. Note that there exist a set $\{\phi_i\}_{i=1}^\infty \subset C^2([0,1],\mathbb{R})$ which is dense in the space $C([0,1],\mathbb{R})$ with the uniform norm. Hence,

$$\Lambda\setminus\Lambda_* = \bigcup_{i=0}^\infty\bigcup_{j=0}^\infty (\Lambda\setminus\Lambda_*(\phi_i,1/j))\,.$$

Thus, the statement follows. □

4.2 Ploughing in the phase space

In this section, we want to study the map $f \in \mathcal{A}$ which satisfies the conditions (CE) and (WR) and obtain some estimates in the phase space. The main results are Propositions 4.22, 4.26 and 4.29. Proposition 4.22 and Proposition 4.26 have been essentially made in § 3. Compared with the notations used in § 3, we introduce the following notations.

For $x \in [0,1]$ and $\varepsilon > 0$, let $0 \le \boldsymbol{S}_1(x,\varepsilon) < \boldsymbol{S}_2(x,\varepsilon) < \ldots < \boldsymbol{S}_n(x,\varepsilon) < \ldots$ be the all integers such that $f^{\boldsymbol{S}_j(x,\varepsilon)}(x) \in \widetilde{B}(\varepsilon)$, and let

$$\boldsymbol{d}_j(x,\varepsilon) = q_\varepsilon(f^{\boldsymbol{S}_j(x,\varepsilon)}(x)). \tag{4.26}$$

Define

$$\boldsymbol{P}_j(x,\varepsilon) = \frac{|Df^{\boldsymbol{S}_j(x,\varepsilon)}(x)|}{\mathrm{dist}(f^{\boldsymbol{S}_j(x,\varepsilon)}(x),\mathcal{C})},$$

$$\boldsymbol{p}_j(x,\varepsilon) = \log\frac{\boldsymbol{P}_j(x,\varepsilon)}{A(x,f,\boldsymbol{S}_j(x,\varepsilon))},$$

and

$$\widetilde{\boldsymbol{p}}_j(x,\varepsilon) = \min\left\{\boldsymbol{p}_j(x,\varepsilon), \boldsymbol{d}_j(x,\varepsilon)\right\}.$$

4.2.1 Essential returns

Definition 4.6. We say that $\boldsymbol{S}_k(x,\varepsilon)$ is an essential return time of x into $\widetilde{B}(\varepsilon)$, if

$$\boldsymbol{P}_k(x,\varepsilon) \geq 3^{k-i}\boldsymbol{P}_i(x,\varepsilon), \text{ for all } 1 \leq i < k.$$

Given $C_0 > 0$ and positive integer n, we define

$$\mathcal{T}_{\text{ess}}(x,\varepsilon) = \{k \geq 1 : \boldsymbol{S}_k(x,\varepsilon) \text{ is an essential return time of } x \text{ into } \widetilde{B}(\varepsilon)\},$$

and

$$\widehat{\mathcal{T}}_{\text{ess}}(C_0,n,x,\varepsilon) = \{k \in \mathcal{T}_{\text{ess}}(x,\varepsilon) : \widetilde{\boldsymbol{p}}_k(x,\varepsilon) > C_0 \text{ and } \boldsymbol{S}_k(x,\varepsilon) < n\}.$$

Note that $\boldsymbol{S}_1(x,\varepsilon)$ is an essential return time. The goal of this section is to prove the following:

Proposition 4.22. *Given $C_0 > 0$, $\tau > 0$, $d > 0$ and $\gamma \in (0,1)$, the following holds provided that $\varepsilon > 0$ is small enough:*

(i) Let $x \in [0,1]$ be such that $\sum_{i=0}^{n-1} q_\varepsilon(f^i(x)) > \tau n$, then we have

$$\sum_{k\in\widehat{\mathcal{T}}_{ess}(C_0,n,x,\varepsilon)} \widetilde{\boldsymbol{p}}_k(x,\varepsilon) \geq \gamma\tau n - C_0 m,$$

where $m = \#\{0 \leq j \leq n-1 : f^j(x) \in \widetilde{B}(\varepsilon)\}$.

(ii) For any $x \in [0,1]$, if

$$\max_{i=0}^{n-1} q_\varepsilon(f^i(x)) \geq d \geq C_0\gamma^{-1},$$

then there exists $k \in \widehat{\mathcal{T}}_{ess}(C_0,n,x,\varepsilon)$ such that $\widetilde{\boldsymbol{p}}_k(x,\varepsilon) > \gamma d$.

Remark. The above definitions and Proposition 4.22 are the analogue of what is done in § 3.2.2. In this proposition, we consider the behavior of the orbit before the n-th iteration instead of the n-th return time which has been investigated in Proposition 3.8.

In the following, fix $\gamma \in (0,1)$ and denote $\rho = 1-\sqrt{\gamma}$, $\rho_1 = \rho/4$, $\rho_2 = \rho_1/(2\ell_{max})$. Let $\varepsilon > 0$ denote a small constant and we fix $x \in [0,1]$. For simplicity, we shall drop ε and x from the notations. So $\boldsymbol{S}_i = \boldsymbol{S}_i(x,\varepsilon)$, $\boldsymbol{d}_i = \boldsymbol{d}_i(x,\varepsilon)$, etc.

Free returns

Define

$$\widehat{\boldsymbol{S}}_i = \sup\{S > \boldsymbol{S}_i : \mathrm{A}(f^{\boldsymbol{S}_i+1}(x), f, S-\boldsymbol{S}_i) \le \theta_0 e^{(\boldsymbol{d}_i-1)\ell(c)}\varepsilon^{-1}\},$$

and

$$\widetilde{\boldsymbol{S}}_i = \inf\{S > \widehat{\boldsymbol{S}}_i : f^S(x) \in \widetilde{B}(\varepsilon)\},$$

where c is the critical point of f which is closest to $f^{\boldsymbol{S}_i}(x)$ and θ_0 is the constant as in Lemma 3.10.

Proposition 4.23.

$$\sum_{k=\boldsymbol{S}_i+1}^{\widehat{\boldsymbol{S}}_i} q_\varepsilon(f^k(x)) \le \rho_1 \cdot \boldsymbol{d}_i. \tag{4.27}$$

Moveover, if $\boldsymbol{S}_j = \widetilde{\boldsymbol{S}}_i$, *then*

$$|Df^{\boldsymbol{S}_j-\boldsymbol{S}_i-1}(f^{\boldsymbol{S}_i+1}(x))| \cdot D_{c_j}(\varepsilon) \ge 3^{j-i}\exp(\ell(c_i)\boldsymbol{d}_i - \rho_1\boldsymbol{d}_i + 1), \tag{4.28}$$

and

$$\log\frac{\boldsymbol{P}_j}{\boldsymbol{P}_i} > \boldsymbol{d}_j - \rho_1\boldsymbol{d}_i + (\log 3)\cdot(j-i), \tag{4.29}$$

where c_i *and* c_j *are the critical points of* f *which are closest to respectively* $f^{\boldsymbol{S}_i}(x)$ *and* $f^{\boldsymbol{S}_j}(x)$.

Proof. Let $a = 2\ell_{max}/(\ell_{min}-1)$, $\varepsilon' = e^a\varepsilon$, $y = f^{\boldsymbol{S}_i+1}(x)$ and $v = f(c_i)$. Note that

$$\mathrm{A}(y, f, \widehat{\boldsymbol{S}}_i - \boldsymbol{S}_i) \le \frac{\theta_0}{|y-v|}.$$

So by Lemma 3.10, for $0 \le k < \widehat{\boldsymbol{S}}_i - \boldsymbol{S}_i$, we have

$$e^{-1}|Df^{k+1}(y)| \le |Df^{k+1}(v)| \le e|Df^{k+1}(y)|, \tag{4.30}$$

and

$$|Df^{k+1}(y)| \ge e^{-1}\frac{|f^{k+1}(v) - f^{k+1}(y)|}{|v-y|}. \tag{4.31}$$

Hence, for each $0 \le k < \widehat{\boldsymbol{S}}_i - \boldsymbol{S}_i$, if $f^k(v) \in \widetilde{B}(\varepsilon')$, we have that

$$\exp\left(\frac{-2}{\ell_{min}-1}\right) \le \frac{\operatorname{dist}(f^k(y),\mathcal{C})}{\operatorname{dist}(f^k(v),\mathcal{C})} \le \exp\left(\frac{2}{\ell_{min}-1}\right) \tag{4.32}$$

provided that $\varepsilon > 0$ is small enough.

We first prove (4.27). We can assume that $\boldsymbol{d}_i > (2a+1) + \log\theta_0$. Otherwise, we have $\widehat{\boldsymbol{S}}_i \le \boldsymbol{S}_{i+1}$, provided that $\varepsilon > 0$ is sufficiently small. Let $T_1 < T_2 < \cdots < T_m < \widehat{\boldsymbol{S}}_i - S_i$ be all the positive integers such that $f^{T_k}(v) \in \widetilde{B}(\varepsilon')$. Then

$$e^{-a-1}\frac{|Df^{T_m}(v)|}{\operatorname{dist}(f^{T_m}(v),\mathcal{C})} \le \frac{|Df^{T_m}(y)|}{\operatorname{dist}(f^{T_m}(y),\mathcal{C})} \le \theta_0 e^{(\boldsymbol{d}_i-1)\ell(c_i)}\varepsilon^{-1}.$$

We have T_m is large, provided that $\varepsilon > 0$ is sufficiently small. Since f satisfies (CE), there exist $\lambda > 1$, such that

$$\frac{|Df^{T_m}(v)|}{\operatorname{dist}(f^{T_m}(v),\mathcal{C})} = \frac{|Df^{T_m+1}(v)|}{|Df(f^{T_m}(v))|\operatorname{dist}(f^{T_m}(v),\mathcal{C})} \ge \frac{\lambda^{T_m+1}}{\varepsilon'}.$$

Hence, we get that

$$T_m + 1 \le \frac{\log\theta_0 + (2a+1) + (\boldsymbol{d}_i - 1)\ell(c_i)}{\log\lambda} \le \frac{2\boldsymbol{d}_i\ell(c_i)}{\log\lambda}.$$

By (4.32), we have $q_\varepsilon(f^k(y)) \le q_{\varepsilon'}(f^k(v))$ for $0 \le k < \widehat{\boldsymbol{S}}_i - \boldsymbol{S}_i$. Since f satisfies the (WR) condition, there exists $\delta > 0$ and $N \in \mathbb{N}$ such that following holds for any $n > N$:

$$\sum_{k=0}^{n-1} \log(|Df(f^k(v))|) \cdot 1_{C(f,\delta)}(f^k(v)) \ge -n \cdot \left(\frac{\rho_1 \log\lambda(\ell_{min}-1)}{2\ell_{max}}\right).$$

Furthermore, for $\varepsilon > 0$ small enough, we have that

$$(\ell_{min}-1) \cdot \sum_{k=0}^{\widehat{\boldsymbol{S}}_i-\boldsymbol{S}_i-1} q_{\varepsilon'}(f^k(v)) \le \sum_{k=0}^{T_m} -\log|Df(f^k(v))| \cdot 1_{C(f,\delta)}(f^k(v)).$$

Thus,

$$\sum_{k=\boldsymbol{S}_i+1}^{\widehat{\boldsymbol{S}}_i} q_\varepsilon(f^k(x)) \le \sum_{k=0}^{\widehat{\boldsymbol{S}}_i-\boldsymbol{S}_i-1} q_{\varepsilon'}(f^k(v)) \le \rho_1 d_i, \tag{4.33}$$

which implies (4.27).

To obtain (4.29) it suffices to prove the following two inequalities:

$$|Df^{\boldsymbol{S}_j-\boldsymbol{S}_i-1}(y)| \geq \Lambda_1(\varepsilon)^{j-i}(D_{c_j}(\varepsilon))^{-1}, \tag{4.34}$$

and

$$|Df^{\boldsymbol{S}_j-\boldsymbol{S}_i-1}(y)| \geq \kappa \exp(\ell(c_i) - \rho_2 \ell_{max} \boldsymbol{d}_i)(D_{c_j}(\varepsilon))^{-1}, \tag{4.35}$$

where $\Lambda_1(\varepsilon) \to \infty$ as $\varepsilon \to 0$ and $\kappa > 0$ is a constant.

Let us prove (4.28). Applying Proposition 2.1, we obtain

$$|Df^{\boldsymbol{S}_j-\boldsymbol{S}_{j-1}-1}(f^{\boldsymbol{S}_{j-1}-\boldsymbol{S}_i}(y))| \geq \Lambda(\varepsilon)/D_{c_j}(\varepsilon).$$

Thus (4.34) holds with $\Lambda_1(\varepsilon) = \Lambda(\varepsilon)$ if $j = i+1$. When $j > i+1$, $\boldsymbol{S}_{j-1} - \boldsymbol{S}_i$ is of the form $T_k + 1$ for some $j - i - 1 \leq k \leq m$, hence we get that

$$|Df^{\boldsymbol{S}_j-\boldsymbol{S}_i}(y)| \geq e^{-1}|Df^{T_k+1}(v)| \geq e^{-1}\lambda^{T_k+1}.$$

$(T_k + 1)/k$ is large, provided that $\varepsilon > 0$ is sufficiently small. Hence, we obtain that (4.34) holds with a suitable choice of $\Lambda_1(\varepsilon)$.

Let us prove (4.35). We may certainly assume $(\ell_{max} - 1)\rho_2 \boldsymbol{d}_i \geq 2$. Let

$$A_k = \frac{|Df^{\boldsymbol{S}_k-\boldsymbol{S}_i-1}(y)|}{\mathrm{dist}(f^{\boldsymbol{S}_k-\boldsymbol{S}_i-1}(y), \mathcal{C})}, \text{ and } A'_k = \frac{|Df^{\boldsymbol{S}_k-\boldsymbol{S}_i-1}(y)|}{|\widetilde{B}(c_k;\varepsilon)|}$$

for $i < k \leq j$. Clearly, $A_k \geq A'_k$. By Proposition 2.1 (i), we have

$$\frac{A'_j}{A_k} = |Df^{\boldsymbol{S}_j-\boldsymbol{S}_k}(f^{\boldsymbol{S}_k}(x))| \frac{\mathrm{dist}(f^{\boldsymbol{S}_k}(x), \mathcal{C})}{|\widetilde{B}(c_j;\varepsilon)|} \geq \Lambda(\varepsilon)^{j-k} \exp\left(-\ell_{max} \sum_{k \leq l < j} \boldsymbol{d}_l\right),$$

which, by (4.33), implies

$$\frac{A'_j}{A_k} \geq \Lambda(\varepsilon)^{j-k} e^{-\rho_2 \ell_{max} \boldsymbol{d}_i}. \tag{4.36}$$

Let $\theta = \theta_0/(2e^{\ell_{max}})$. We distinguish two cases.

Case 1. Assume $\mathrm{A}(y, f, \boldsymbol{S}_j - \boldsymbol{S}_i - 1) \geq \theta e^{\boldsymbol{d}_i \cdot \ell(c_i)} \varepsilon^{-1}$. Then by Lemma 3.9, we have

$$\sum_{k=i+1}^{j-1} A_k + A'_j \geq \frac{1}{1+\kappa(\varepsilon)} \mathrm{A}(y, f, \boldsymbol{S}_j - \boldsymbol{S}_i - 1) \geq \theta e^{\boldsymbol{d}_i \cdot \ell(c_i)}(2\varepsilon)^{-1}.$$

Together with (4.36), this implies $A'_j \geq \theta \exp(\ell(c_i)\boldsymbol{d}_i - \rho_2 \ell_{max} \boldsymbol{d}_i)(4\varepsilon)^{-1}$, provided that $\varepsilon > 0$ is small enough. Thus (4.35) holds in this case.

Case 2. Assume $\mathrm{A}(y, f, \boldsymbol{S}_j - \boldsymbol{S}_i - 1) < \theta e^{\ell(c_i)\cdot \boldsymbol{d}_i} \varepsilon^{-1}$. In particular we have $\boldsymbol{S}_j - 1 \leq \widehat{\boldsymbol{S}}_i$ which implies $\widehat{\boldsymbol{S}}_i = \boldsymbol{S}_j - 1$. By maximality of $\widehat{\boldsymbol{S}}_i$ we have

$$A_j = \mathrm{A}(y, f, \boldsymbol{S}_j - \boldsymbol{S}_i) - \mathrm{A}(y, f, \widehat{\boldsymbol{S}}_i - \boldsymbol{S}_i) \geq \theta e^{\ell(c_i)\cdot \boldsymbol{d}_i} \varepsilon^{-1}.$$

So (4.35) holds if $\boldsymbol{d}_j \leq \rho_2 \ell_{max} \boldsymbol{d}_i$. Assume $\boldsymbol{d}_j > \rho_2 \ell_{max} \boldsymbol{d}_i$. By (4.33),

$$q_{\varepsilon'}(f^{\boldsymbol{S}_j - \boldsymbol{S}_i - 1}(v)) \leq \rho_2 \boldsymbol{d}_i \leq \rho_2 \ell_{max} \boldsymbol{d}_i - 2.$$

Thus there exists a constant $\kappa_1 > 0$ such that

$$|f^{\boldsymbol{S}_j - \boldsymbol{S}_i - 1}(v) - f^{\boldsymbol{S}_j - \boldsymbol{S}_i - 1}(y)| \geq \kappa_1 e^{-\rho_2 \ell_{max} \boldsymbol{d}_i} |\widetilde{B}(c_j; \varepsilon)|$$

Thus, by (4.31),

$$|Df^{\boldsymbol{S}_j - \boldsymbol{S}_i - 1}(y)| \geq e^{-1} \frac{|f^{\boldsymbol{S}_j - \boldsymbol{S}_i - 1}(v) - f^{\boldsymbol{S}_j - \boldsymbol{S}_i - 1}(y)|}{|v - y|} \geq \frac{\kappa_1 \exp(\ell(c_i)\boldsymbol{d}_i - \rho_2 \ell_{max} \boldsymbol{d}_i)}{D_{c_j}(\varepsilon)}.$$

So the inequality (4.35) holds.

From (4.28), we can imply (4.29) directly. □

We define positive integers $i_1 < i_2 < \cdots$ in the following way: $i_1 = 1$. Once i_k and $\boldsymbol{S}_{i_k}$ are both well-defined, let i_{k+1} be such that $\widetilde{\boldsymbol{S}}_{i_k} = \boldsymbol{S}_{i_{k+1}}$. The procedure stops whenever $\boldsymbol{S}_{i_k}$ or $\widetilde{\boldsymbol{S}}_{i_{k+1}}$ is not well-defined. The integers $\boldsymbol{S}_{i_k}$, $k = 1, 2, \cdots$ are called *free return times* of x into $\widetilde{B}(\varepsilon)$. Compared with Lemmas 3.12 and 3.13, we obtain the following lemmas in the phase space. The proofs of these two lemmas are the same as the ones of Lemmas 3.12 and 3.13.

Lemma 4.24. *An essential return time is a free return time.*

Lemma 4.25. *If $\boldsymbol{S}_i < \boldsymbol{S}_j$ are consecutive essential return times of x into $\widetilde{B}(\varepsilon)$, then*

$$\sum_{i<k<j} \boldsymbol{d}_k \leq \rho \boldsymbol{d}_i, \tag{4.37}$$

and

$$\boldsymbol{p}_j \geq \boldsymbol{d}_j - \rho \boldsymbol{d}_i. \tag{4.38}$$

Proof of Proposition 4.22. (i) By hypothesis, we have that $\sum_{k=1}^m \boldsymbol{d}_k \geq \tau n$. Let $1 = i_1 < i_2 < \cdots < i_p \leq m$ be all the integers such that $\boldsymbol{S}_{i_j}$ is an essential return time of x into $\widetilde{B}(\varepsilon)$. For convenience of notations, put $i_0 = 0$ and $\boldsymbol{d}_{i_0} = 0$.

By (4.37) in Lemma 4.25, we have

$$\sum_{j=1}^{p} \boldsymbol{d}_{i_j} \geq (1-\rho) \sum_{j=1}^{m} \boldsymbol{d}_j \geq (1-\rho)\tau n.$$

Thus

$$\sum_{j=1}^{p} (\boldsymbol{d}_{i_j} - \rho \boldsymbol{d}_{i_{j-1}}) \geq (1-\rho)^2 \tau n = \gamma \tau n.$$

By (4.38) in Lemma 4.25, for each $2 \leq j \leq p$ we have

$$\boldsymbol{p}_{i_j} \geq \boldsymbol{d}_{i_j} - \rho \boldsymbol{d}_{i_{j-1}}.$$

By Lemma 3.9, this estimate is also true for $j = 1$. Thus

$$\widetilde{\boldsymbol{p}}_{i_j} \geq \boldsymbol{d}_{i_j} - \rho \boldsymbol{d}_{i_{j-1}}$$

holds for all $j = 1, 2, \ldots, p$, which implies

$$\sum_{j=1}^{p} \widetilde{\boldsymbol{p}}_{i_j} \geq \sum_{j=1}^{p} (\boldsymbol{d}_{i_j} - \rho \boldsymbol{d}_{i_{j-1}}) \geq \gamma \tau n.$$

Consequently,

$$\sum_{k \in \widehat{\mathcal{T}}_{\mathrm{ess}}(C_0, n, x, \varepsilon)} \widetilde{\boldsymbol{p}}_k \geq \sum_{j=1}^{p} \widetilde{\boldsymbol{p}}_{i_j} - C_0 p \geq \gamma \tau n - C_0 m.$$

(ii) Let k be the minimal integer such that $\boldsymbol{d}_k = \max_{i=0}^{n-1} q_\varepsilon(f^i(x))$. Then we have that $\boldsymbol{S}_k \leq n - 1$ and $\boldsymbol{d}_j < \boldsymbol{d}_k$ holds for any $1 \leq j < k$. By (4.37) in Lemma 4.25, it follows that $\boldsymbol{S}_k$ is an essential return time of x into $\widetilde{B}(\varepsilon)$ and hence

$$\boldsymbol{p}_k \geq (1-\rho)\boldsymbol{d}_k > \gamma d > C_0.$$

The statement is proved. $\square$

4.2.2 Tail estimates

With the essential return argument, we can obtain the following result.

Proposition 4.26. *(i) Given $\tau > 0$, for each $\varepsilon > 0$ small, we have*

$$\left|\left\{x : \sum_{i=0}^{n-1} q_\varepsilon(f^i(x)) \geq \tau n\right\}\right| \leq e^{-\tau n/8}$$

holds for n large enough.

(ii) Given $a > 0$, for each ε small enough, we have

$$\left|\left\{x : \max_{i=0}^{n-1} q_\varepsilon(f^i(x)) > a \log n\right\}\right| \leq \frac{1}{n^{a-2}}$$

holds for n large enough.

Remark. In this proposition, we estimate the measure of the points which have not satisfied the required condition at time n in the phase space. In § 3.3, we have obtained the similar result (Reduced Theorem A), in terms of return time, in the parameter space, with the help of the parameter map $\xi_n^{(c)}(t) = f_t^{n+1}(c)$.

The strategy here is similar to the one in § 3.3. More precisely, we construct a special family which has been defined in § 3.3 and prove the above sets are contained in the special family of balls with large total depth.

We first recall the notations and definitions about the special family.

Given $B = B(a_0, r)$ and $x \in \mathbb{R}$, we define

$$\operatorname{dep}(x|B) = \begin{cases} \inf\{k \in \mathbb{N} : |x - a_0| \geq e^{-k} r\}, & \text{if } |x - a_0| < e^{-2} r; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, for each $k \in \mathbb{N}$, let

$$B^{(k)} = B(a_0, e^{-k} r). \tag{4.39}$$

A countable family $\mathcal{M} = \{B_i = B(a_i, r_i)\}_{i \in \mathcal{I}}$ is called *special* if the following holds: For any $i, j \in \mathcal{I}$, if $a_i \in B_j^{(1)}$ then there exists $k = k(i,j) \geq 1$ such that $B_i \subset B_j^{(k-1)} \setminus B_j^{(k+1)}$. In particular, the centers $a_i, i \in \mathcal{I}$ are pairwise distinct.

Given a special family as above, define a partition of $\mathcal{I}$ into sets $\mathcal{I}_0, \mathcal{I}_1, \ldots,$

inductively as follows. Put

$$\mathcal{I}_0 = \{i \in \mathcal{I} : \text{ for any } j \in \mathcal{I}, j \neq i, \text{ we have } a_i \notin B_j^{(1)}\},$$

and for each $k \geq 1$, let

$$\mathcal{I}_k = \left\{ i \in \mathcal{I} \setminus \bigcup_{m=0}^{k-1} \mathcal{I}_m : \text{ for any } j \in \mathcal{I} \setminus \bigcup_{m=0}^{k-1} \mathcal{I}_m, j \neq i, \text{ we have } a_i \notin B_j^{(1)} \right\}.$$

The minimal integer $n \geq 0$ for which $\mathcal{I}_n = \emptyset$, if any, is called the *height* of $\mathcal{M}$. The support of $\mathcal{M}$ is defined as the union of all the elements of $\mathcal{M}$.

Then we construct a special family and apply Lemma 3.16 in § 3.3 to prove Proposition 4.26.

A ball $B(a_0, r)$ is called λ*-bounded of order* k*,* if for each $0 \leq j \leq k$, we have

$$\sup_{x,y \in B(a_0,r)} \frac{|Df^j(x)|}{|Df^j(y)|} \leq \lambda.$$

For $k \geq 0$, let $\mathscr{C}_k$ denote the set of $x \in [0,1]$ for which the following hold:

- $f^k(x) \in \mathcal{C}$;
- $f^j(x) \cap \mathcal{C} = \emptyset$ for $0 \leq j \leq k-1$.

A ball $B(a_0, r)$ is called *pre-critical of order* k*,* if $a_0 \in \mathscr{C}_k$.

For $k \geq 0$, $\lambda > 1$ and $a_0 \in \mathscr{C}_k$, let $r_\lambda(a_0, \varepsilon)$ be the maximal number r which satisfies the following properties:

(i) $f^k(B(a_0, r)) \subset \widetilde{B}(\varepsilon)$;

(ii) $B(a_0, r)$ is λ-bounded of order k.

Given a positive integer m and $\lambda > 1$, let

$$\mathcal{M}_{m,\lambda}(\varepsilon) = \left\{ B(a_0, r_\lambda(a_0, \varepsilon)) \mid \begin{array}{c} a_0 \in \mathscr{C}_k \text{ for some } k \geq 0 \text{ and} \\ \text{there exists } x \in B(a_0, r_\lambda(a_0, \varepsilon)) \text{ such that} \\ \#\{0 \leq j \leq k : f^j(x) \in \widetilde{B}(\varepsilon)\} \leq m. \end{array} \right\}.$$

Lemma 4.27. *There exists* $\lambda \in (1, e)$ *such that for each* $m > 1$ *and each* $\varepsilon > 0$ *small,* $\mathcal{M}_{m,\lambda}(\varepsilon)$ *is a special family of height at most* m*.*

Proof. Assume $\lambda > 1$ is very close to 1. To prove that $\mathcal{M}_{m,\lambda}(\varepsilon)$ is special, let $B_i = B(a_i, r_i)$ and $i = 1, 2$ be distinct balls in $\mathcal{M}_{m,\lambda}(\varepsilon)$ of order k_i, such that $a_1 \in B_2^{(1)}$. We need to prove $|B_1|/|a_1 - a_2|$ is small. Let c_1, $c_2 \in \mathcal{C}$ be such that $f^{k_i}(a_i) = c_i$. Since $f^j(a_1) \notin \mathcal{C}$ for each $0 \leq j \leq k_2$, we have $k_1 > k_2$. By the bounded distortion property of $f^{k_2}|B_1$ and $f^{k_2}|B_2$, it suffices to show that

$$\sup_{x \in B_1} \frac{|f^{k_2}(x) - f^{k_2}(a_1)|}{|f^{k_2}(a_1) - c_2|}$$

is sufficiently small. This is clear: for each $x \in B_1$, and for each $0 \leq j \leq k_2 + 1 \leq k_1$,

$$\lambda^{-1}|Df^j(x)| \leq |Df^j(a_1)| \leq \lambda |Df^j(x)|,$$

hence, $\lambda^{-2}|Df(f^{k_2}(x))| \leq |Df(f^{k_2}(a_1))| \leq \lambda^2 |Df(f^{k_2}(x))|$, so the statement follows from the local behavior of f near c_2.

Let us prove the height of $\mathcal{M}_{m,\lambda}(\varepsilon)$ does not exceed m. Otherwise, there would exist $B_j \in \mathcal{M}_{m,\lambda}(\varepsilon)$, $0 \leq j \leq m$, such that $B_m \subsetneq B_{m-1} \subsetneq \cdots \subsetneq B_0$. Let k_j be the order of B_j. Then as above, we would have $k_0 < k_1 < \cdots < k_m$. Then for $x \in B_m$, $\{0 \leq j \leq k_m : f^j(x) \in \widetilde{B}(\varepsilon)\} \supset \{k_0, k_1, \ldots, k_m\}$ would contain at least $m + 1$ elements, a contradiction. $\square$

Now, we fix a constant $\lambda \in (1, e)$ so that the conclusion of Lemma 4.27 holds. Then we have

Lemma 4.28. *There exists $C_0 > 0$ such that following holds provided that $\varepsilon > 0$ is small enough. For any $x \in [0, 1]$, if there exists $1 \leq j \leq m$ such that $\widetilde{\boldsymbol{p}}_j(x, \varepsilon) > C_0$, then there is a pre-critical ball $B(a, r)$ of order $\boldsymbol{S}_j(x, \varepsilon)$ in $\mathcal{M}_{m,\lambda}(\varepsilon)$ such that*

$$dep(x|B(a, r)) \geq \widetilde{\boldsymbol{p}}_j(x, \varepsilon) - C_0.$$

Proof. Fix $x \in [0, 1]$ and a small constant $\varepsilon > 0$, let $\boldsymbol{S}_j = \boldsymbol{S}_j(x, \varepsilon)$, $\boldsymbol{p}_j = \boldsymbol{p}_j(x, \varepsilon)$ and $\boldsymbol{d}_j = \boldsymbol{d}_j(x, \varepsilon)$. By Lemma 3.10, we know that $B(x, r_0)$ is a λ-bounded ball of order $\boldsymbol{S}_j$, where

$$r_0 = \frac{\theta_0 \log \lambda}{A(x, f, \boldsymbol{S}_j)}.$$

Assume that $\boldsymbol{p}_j$ and $\boldsymbol{d}_j$ are large. Since $|Df^{\boldsymbol{S}_j}(x)| \cdot r_0 \asymp e^{\boldsymbol{p}_j} \cdot \mathrm{dist}(f^{\boldsymbol{S}_j}(x), \mathcal{C})$, by the bounded distortion property of $f^{\boldsymbol{S}_j}|B(x, r_0)$, there exists a λ-bounded ball $B(a, r_*) \subset B(x, r_0)$ of order $\boldsymbol{S}_j$ such that $r_* \asymp r_0$, $a \in \mathscr{C}_{\boldsymbol{S}_j}$ and $\mathrm{dep}(x|B(a, r_*)) - \boldsymbol{p}_j$ is bounded away from $-\infty$. Let $r = r_\lambda(a,\varepsilon)$. Clearly, $B(a,r) \in \mathcal{M}_{m,\lambda}(\varepsilon)$. If $r \geq r_*$, then $\mathrm{dep}(x|B(a,r)) \geq \mathrm{dep}(x|B(a,r_*))$ and we are done. So assume $r < r_*$

Then $\partial(f^{\boldsymbol{S}_j}(B(a,r))) \cap \partial\widetilde{B}(f^{\boldsymbol{S}_j}(a);\varepsilon) \neq \emptyset$. It follows $f^{\boldsymbol{S}_j}(B(a,r)) \supset \widetilde{B}(f^{\boldsymbol{S}_j}(a);\varepsilon')$ holds for some $\varepsilon' \asymp \varepsilon$. Since $|f^{\boldsymbol{S}_j}(x) - f^{\boldsymbol{S}_j}(a)| \geq e^{-\boldsymbol{d}_j}|\widetilde{B}(f^{\boldsymbol{S}_j}(a);\varepsilon)|$, we conclude that $\mathrm{dep}(x|B(a,r)) - \boldsymbol{d}_j$ is bounded away from $-\infty$. The lemma is proved. $\square$

Proof of Proposition 4.26. (i) Let $\lambda > 1$ and $C_0 > 0$ be as above. For any $x \in [0,1]$, positive integer n and $\varepsilon > 0$, let

$$m(x,n;\varepsilon) = \#\{0 \leq k \leq n-1 : f^k(x) \in \widetilde{B}(\varepsilon)\} \text{ and } m = \sup_{x\in[0,1]} m(x,n;\varepsilon).$$

Then n/m is large, when ε is small enough and n is large. If $\sum_{i=0}^{n-1} q_\varepsilon(f^i(x)) > \tau n$, by Proposition 4.22 (i) (taking $\gamma = 1/2$) and Lemma 4.28, we have

$$\sum_{B\in\mathcal{M}_{m,\lambda}(\varepsilon)} \mathrm{dep}(x|B) \geq \sum_{k\in\widehat{\mathcal{T}}_{\mathrm{ess}}(C_0,n,x,\varepsilon),} \left(\widetilde{\boldsymbol{p}}_k(x,\varepsilon) - C_0\right) \geq \frac{1}{2}\tau n - 2C_0 m.$$

By Lemma 4.27 and Lemma 3.16 (taking $\kappa = 1/2$), it follows that

$$\left|\left\{x : \sum_{i=0}^{n-1} q_\varepsilon(f^i(x)) \geq \tau n\right\}\right| \leq K^m e^{C_0 m - \tau n/4} \leq e^{-\tau n/8}$$

holds for n large enough. The statements follows.

(ii) For any $a > 0$, choose $\gamma < 1$ close to 1. Given n large enough, consider $x \in [0,1]$ with

$$\max_{i=0}^{n-1} q_\varepsilon(f^i(x)) > a\log n > C_0\gamma^{-1}.$$

By Proposition 4.22 (ii), there exists $m \in \widehat{\mathcal{T}}_{\mathrm{ess}}(C_0,n,x,\varepsilon)$ such that

$$\widetilde{\boldsymbol{p}}_m(x,\varepsilon) > a\gamma\log n > C_0.$$

By Lemma 4.28, there exist an integer $0 \leq k < n$ and $b \in \mathscr{C}_k$ such that

$$\mathrm{dep}(x|B(b,r_\lambda(b,\varepsilon))) > a\gamma\log n - C_0.$$

For each k, the family $\{B(b, r_\lambda(b,\varepsilon))\}_{b\in\mathscr{C}_k}$ is pairwise disjoint, it follows that

$$\left|\left\{x : \max_{i=0}^{n-1} q_\varepsilon(f^i(x)) > a\log n\right\}\right| \le e^{-a\gamma\log n + C_0}\cdot n < \frac{1}{n^{a-2}}.$$

□

4.2.3 Summability control

Proposition 4.29. *Given $\varepsilon > 0$ small enough, there exists $\rho(\varepsilon) > 0$ such that for each $x \in [0,1]$ and each positive integer n, we have*

$$\sum_{k=0}^{n} |Df^k(x)|^{-1} \le \rho(\varepsilon)\exp\Big(\ell_{max}\cdot \max_{k=0}^{n-1} q_\varepsilon(f^k(x))\Big).$$

Proof. Fix $\varepsilon > 0$ small enough. Let $0 \le S_1 < S_2 < \cdots < S_m < n$ be all the integers such that $f^{S_k}(x) \in \widetilde{B}(\varepsilon)$, let $d_k = q_\varepsilon(f^{S_k}(x))$ and let c_k denote the critical point of f which is closest $f^{S_k}(x)$. Suppose S_i and S_j are consecutive free return times of x into $\widetilde{B}(\varepsilon)$. By the definition of free return time, we obtain that

$$\mathrm{A}(f^{S_i+1}(x), f, S_{j-1}-S_i) \le \theta_0 e^{(d_i-1)\ell(c_i)}\varepsilon^{-1} \le \frac{\theta_0}{|f^{S_i+1}(x) - f(c_i)|}.$$

By Lemma 3.10 and the hypothesis that f satisfies the condition (CE), there exist constants $C > 0$ and $\lambda > 1$ such that

$$|Df^k(f^{S_i+1}(x))| \ge e^{-1}|Df^k(f(c_i))| \ge e^{-1}C\lambda^k$$

holds for any $0 \le k \le S_{j-1} - S_i$, which implies that

$$\sum_{i<k<j} |Df^{S_k+1}(x)|^{-1} \le \frac{e}{(\lambda-1)C}|Df^{S_i+1}(x)|^{-1}. \tag{4.40}$$

By (4.28) in Proposition 4.23, we obtain that

$$\frac{|Df^{S_j+1}(x)|}{|Df^{S_i+1}(x)|} \ge 3^{j-i}\exp\Big((\ell(c_i)-1)d_i - (\ell(c_j)-1)d_j\Big). \tag{4.41}$$

Let $0 \le S_{i_1} < S_{i_2} < \cdots S_{i_p} < n$ be the all free return times of x into $\widetilde{B}(\varepsilon)$. By

(4.41), we have that

$$|Df^{S_{i_k}+1}(x)| \geq 3^{i_k - i_1} \exp\Big(-(\ell(c_{i_k}) - 1)d_{i_k}\Big)|Df^{S_1}(x)|D_{c_1}(\varepsilon). \tag{4.42}$$

Together with (4.40) and Proposition 2.1 (ii), there exist constant $C_1 > 0$ and $\widehat{\rho}(\varepsilon) > 0$ such that

$$\begin{aligned}\sum_{k=1}^{m} |Df^{S_k+1}(x)|^{-1} &\leq C_1 \sum_{k=1}^{p} |Df^{S_{i_k}+1}(x)|^{-1} \\ &\leq \widehat{\rho}(\varepsilon) \exp\Big(\ell_{max} \cdot \max_{k=0}^{n-1} q_\varepsilon(f^k(x))\Big).\end{aligned}$$

Let

$$W_0 = \sup_{c \in \mathcal{C}} \sum_{n=0}^{\infty} |Df^n(f(c))|^{-1}.$$

By Proposition 3.3, we have

$$\sum_{k=S_i+1}^{S_{i+1}} |Df^k(x)|^{-1} \leq |Df^{S_i+1}(x)|^{-1} \cdot W_0,$$

for each $i = 1, \cdots, m-1$ and

$$\sum_{k=S_m+1}^{n} |Df^k(x)|^{-1} \leq |Df^{S_m+1}(x)|^{-1} \cdot W_0.$$

Thus, by Proposition 2.1 (ii), there exists $\rho(\varepsilon) > 0$ such that

$$\begin{aligned}\sum_{k=0}^{n} |Df^k(x)|^{-1} &\leq \sum_{k=0}^{S_1} |Df^k(x)|^{-1} + \sum_{k=1}^{m} |Df^{S_k+1}(x)|^{-1} \cdot W_0 \\ &\leq \rho(\varepsilon) \exp\Big(\ell_{max} \cdot \max_{k=0}^{n-1} q_\varepsilon(f^k(x))\Big).\end{aligned}$$

$\square$

4.3 Harvest in the parameter space

As what we have done in § 3, we will transfer the estimates in the phase space to the parameter space and prove the Reduced Main Theorem B*. So let (f_t), ϕ, δ, Θ,

μ_t and $z(t)$ be as in the Reduced Theorem B*. Without loss of generality, we can assume that $t_0 = 0$ and let $f = f_0$.

4.3.1 Continuity of acip

Proposition 4.30. *There exists $\omega > 0$ such that*

$$\left| \int \phi d\mu_t - \int \phi d\mu_0 \right| < \frac{\delta}{4} \tag{4.43}$$

holds for any $t \in [-\omega, \omega] \cap \Lambda_\Theta(\phi, \delta)$.

Proof. We argue by contradiction. Assume that the proposition does not hold. Then there exists a sequence $\{t_n\}_{n=1}^\infty \subset \Lambda_\Theta(\phi, \delta)$ such that $t_n \to 0$ as $n \to \infty$ and such that

$$\left| \int \phi d\mu_{t_n} - \int \phi d\mu_0 \right| \geq \frac{\delta}{4} \tag{4.44}$$

holds for each positive integer n. Since $\{\mu_{t_n}\}_{n=1}^\infty$ is pre-compact in the weak star topology, replacing $\{\mu_{t_n}\}_{n=1}^\infty$ by a subsequence, if necessary, we can assume that the limit $\mu := \lim_{n\to\infty} \mu_{t_n}$ exists.

To complete the proof, we only need to show that

$$\mu = \mu_0. \tag{4.45}$$

Indeed, if $\mu = \mu_0$, we obtain that

$$\lim_{n\to\infty} \int \phi d\mu_{t_n} = \int \phi d\mu_0, \tag{4.46}$$

which contradicts (4.44).

Let us prove (4.45). Since f has a unique acip, we only need to show that μ is an acip for f. By the definitions of $\Lambda_\Theta(\phi, \delta)$ and μ, we obtain that μ is a probability measure which is absolutely continuous with respect to Lebesgue measure. To this end, we need to prove that μ is f-invariant. For any $g \in C([0,1], \mathbb{R})$ and $\varepsilon > 0$, there exists $\sigma > 0$ such that $|g(x) - g(y)| < \varepsilon$ holds for any $|x - y| < \sigma$. For n large enough, we have $|f_{t_n}(x) - f(x)| < \sigma$ holds for any $x \in [0,1]$, which implies that

$$\left| \int g \circ f_{t_n} d\mu_{t_n} - \int g \circ f d\mu_{t_n} \right| < \varepsilon. \tag{4.47}$$

Besides, we have that

$$\int g d\mu = \lim_{n\to\infty} \int g d\mu_{t_n} = \lim_{n\to\infty} \int g \circ f_{t_n} d\mu_{t_n},$$

and

$$\int g \circ f d\mu = \lim_{n\to\infty} \int g \circ f d\mu_{t_n}.$$

Combining these with (4.47), we obtain that μ is f-invariant. The statement follows. $\square$

4.3.2 Stable branches

Given $\omega \in (0,1]$, a family $\mathscr{P} = \{\mathcal{P}(t)\}_{t\in[-\omega,\omega]}$ of finite sets is called ω-*admissible*, if the following holds:

(1) there exist an integer $m \geq 2$, and $P_i \in C^1([-\omega,\omega],[0,1])$, $i = 1,2,\ldots,m$, such that $0 < P_1(t) < P_2(t) < \ldots < P_m(t) < 1$ and $\mathcal{P}(t) = \{P_i(t) : 1 \leq i \leq m\}$ holds for any $t \in [-\omega,\omega]$;

(2) for any $t \in [-\omega,\omega]$, $\{P_i(t)\}_{i=1}^m$ is a f_t-forward invariant set.

Given an ω-admissible family $\mathscr{P}$, for convenience of notations, we regard $P_0(t) \equiv 0$ and $P_{m+1}(t) \equiv 1$. We define

$$\|\mathscr{P}\| := \sup_{i,t} |P_i(t) - P_{i+1}(t)|,$$

where the supremum is taken over all $i \in \{0,\ldots,m\}$ and $t \in [-\omega,\omega]$. Then $\|\mathscr{P}\|$ is called the *size of* $\mathscr{P}$.

Lemma 4.31. *For any $\zeta > 0$, there exist $\omega > 0$ such that there is an ω-admissible family $\mathscr{P}$ with size less than ζ.*

Proof. For any $\zeta > 0$, by Corollary 4.10, there exists $\{p_i\}_{i=1}^m \subset \mathrm{PPer}(f) \cap (0,1)$ such that $\{p_i\}_{i=1}^m$ is a f-forward invariant set and such that $|p_i - p_{i+1}| \leq \zeta/2$ holds for any $i = 0,1,\ldots,m$, where $p_0 = 0$ and $p_{m+1} = 1$.

For each $i \in \{1,2\ldots,m\}$, we will construct a differentiable function P_i with $P_i(0) = p_i$ and such that $\{P_i(t)\}_{i=1}^m$ is f_t-forward invariant for $|t|$ small enough. Let r be the minimal integer such that $y_r := f^r(p_1)$ is a periodic point of f, and let p denote the minimal period of y_r. Since $f \in \mathcal{A}$, by the implicit function theorem, there exist an open set U containing 0, a open set V containing y_r and a differentiable function $y_r(t) : U \to V$ such that $f_t^p(y_r(t)) = y_r(t)$ holds for any $t \in U$. Let $Orb(y_r)$ denote the f-forward orbit of y_r. Then we can define $P_i(t)$ for any i with $\omega(p_i) = Orb(y_r)$. Indeed, if $p_i \in Orb(y_r)$, then there exists $1 \le j \le p$ such that $p_i = f^j(y_r)$. In this case, we can define $P_i(t) = f_t^j(y_r(t))$. Otherwise, let k be the minimal integer such that $f^k(p_i) \in Orb(y_r)$. Then there exist $1 \le j \le p$ such that $f^k(p_i) = f^j(y_r)$. In this case, we let $P_i(t)$ be the differentiable function such that $f_t^k(P_i(t)) = f_t^j(y_r(t))$ holds for $|t|$ small. The implicit function theorem and $\mathrm{PPer}(f) \cap \mathcal{C} = \emptyset$ guarantee the existence of $P_i(t)$. Continue this process for the rest p_i and let $|t|$ be small. The statement follows. □

Given an ω-admissible family $\mathscr{P} = \{\mathcal{P}(t)\}_{t \in [-\omega,\omega]}$, we can define a C^1 parameter family $P(x,t) : [0,1] \times [-\omega,\omega] \to [0,1]$ with respect to $\mathscr{P}$ as follows:

$$P(x,t) = \left(\frac{P_i(t) - P_{i+1}(t)}{P_i(0) - P_{i+1}(0)}\right) \cdot (x - P_i(0)) + P_i(t), \tag{4.48}$$

where i is the integer such that $x \in [P_i(0), P_{i+1}(0)]$.

Definition 4.7. Given $k \in \mathbb{N}$, $\omega > 0$, $\sigma \in (0,\omega)$, $\lambda > 1$ and an ω-admissible family $\mathscr{P}$, we say that a point $x \in [0,1]$ is a λ-bounded σ-stable point of order k with respect to $\mathscr{P}$, if there exists $x(\cdot) \in C^1([-\sigma,\sigma],[0,1])$ with $x(0) = x$ such that the following hold:

- for any $t \in [-\sigma,\sigma]$, $f_t^k(x(t)) = P(f^k(x), t)$, where P is as in (4.48);
- for any $t \in [-\sigma,\sigma]$ and $i \in \{0,1,\dots,k\}$, putting $X_i(t) := f_t^i(x(t))$, we have

$$|\partial_t X_i(t)| \le \lambda \tag{4.49}$$

and

$$1 - \lambda^{-1} < \frac{|Df_t^i(x(t))|}{|Df^i(x)|} < 1 + \lambda^{-1}. \tag{4.50}$$

Lemma 4.32. *There exists a constant $K > 100$ that only depends on (f_t) such that for each $\omega > 0$, each $\varepsilon > 0$ small and each ω-admissible family $\mathscr{P}$, the following holds provided that N is large enough. Let $x \in [0,1]$ and $k \le N$ be such that*

$$\max_{i=0}^{k-1} q_\varepsilon(f^i(x)) < 10 \log N, \tag{4.51}$$

Then x is a N^K-bounded N^{-4K}-stable point of order k with respect to $\mathscr{P}$.

Proof. Let $K = 200 \cdot (\ell_{max})^2$, $\sigma = N^{-4K}$ and $\lambda = N^K$. Given $\omega > 0$, $\varepsilon > 0$ and $\mathscr{P}$, we prove that x is a λ-bounded σ-stable point of order k with respect to $\mathscr{P}$, provided that N is large enough.

Let us prove the statement by induction on k. The case $k = 0$ is clear: let $x(t) = P(x,t)$, then $|\partial_t x(t)| = |\partial_t P(x,t)| \leq \lambda$, provided that N is large enough. Assume the statement holds for $k = j$. Let us consider the case $k = j+1 \leq N$. Since $Df^{j+1}(x) \neq 0$, by the implicit function theorem, there exist an open interval U containing 0, an open interval V containing x and a unique differentiable function $x(\cdot) : U \to V$ with $x(0) = x$ and $f_t^{j+1}(x(t)) = P(f^{j+1}(x), t)$. Let $t_0 \leq \sigma$ be the maximal real number such that the following hold: there exists $x(\cdot) \in C^1([-t_0, t_0], [0,1])$ such that

$$f_t^{j+1}(x(t)) = P(f^{j+1}(x), t) \tag{4.52}$$

and

$$\max_{i=0}^{j} q_\varepsilon(X_i(t)) \leq 11 \log N, \tag{4.53}$$

where $X_i(t) = f_t^i(x(t))$.

Denote $y = f(x)$, by induction hypothesis, there exists $y(\cdot) \in C^1([-\sigma, \sigma], [0,1])$ with $y(0) = y$ and $f_t^j(y(t)) = P(f^{j+1}(x), t)$. By uniqueness, we obtain that $y(t) = X_1(t)$ holds for any $t \in [-t_0, t_0]$. Hence,

$$|Df_t^i(X_1(t))| \geq (1 - \lambda^{-1})|Df^i(y)| > \frac{1}{2}|Df^i(y)| \tag{4.54}$$

holds for any $i = 0, 1, \ldots, j$ and $t \in [-t_0, t_0]$. Differentiate (4.52) with respect to t, we obtain that

$$Df_t^{j+1}(x(t)) \cdot \partial_t x(t) + \sum_{i=0}^{j} \frac{\partial_t F(X_i(t), t)}{Df_t^i(X_1(t))} \cdot Df_t^j(X_1(t)) = \partial_t P(f^{j+1}(x), t), \tag{4.55}$$

which implies that

$$|\partial_t x(t)| \leq \frac{|\partial_t P(f^{j+1}(x), t)|}{|Df_t^{j+1}(x(t))|} + \sum_{i=0}^{j} \frac{|\partial_t F(X_i(t), t)|}{|Df_t^i(X_1(t))|} \cdot \frac{1}{|Df_t(x(t))|}. \tag{4.56}$$

Note that there exists $C_1 > 0$ independent of N such that $|\partial_t P| \leq C_1$. Together with (4.54), we obtain that

$$|\partial_t x(t)| \le \frac{2C_1+2}{|Df_t(x(t))|} \cdot \sum_{i=0}^{j} |Df^i(y)|^{-1}.$$

Combining this, Proposition 4.29 and (4.53), provided that N is large enough, for any $t \in [-t_0, t_0]$, we have $|\partial_t x(t)| \le \lambda$. Together with the induction hypothesis, (4.49) holds for any $i = 0, \dots, j+1$ and $t \in [-t_0, t_0]$.

We then prove that $t_0 = \sigma$. Otherwise, either $t_* = t_0$ or $t_* = -t_0$ is such that

$$\max_{i=0}^{j} q_\varepsilon(X_i(t_*)) \ge 11 \log N.$$

Notice that for any $i \le j$, we have

$$|X_i(t_*) - X_i(0)| \le \lambda \cdot |t_*| \le N^{-3K}.$$

It is a contradiction with (4.51).

To end, let us prove (4.50) for $0 \le i \le j+1$. Since F is a normalized regular family, there exists a constant $C_2 > 0$ such that

$$|\partial_t \partial_x F(x,t)| \le C_2 |Df_t(x)|,$$

for all (x,t). By non-flatness of the critical points, it follows that there exists $C_3 > 0$ such that

$$\frac{|D^2 f_t(X_m(t))|}{|Df_t(X_m(t))|} \le \frac{C_3}{\mathrm{dist}(X_m(t), \mathcal{C})}.$$

Then we obtain that

$$|\partial_t Df_t(X_m(t))| = \left|\partial_t \partial_x F(X_m(t), t) + D^2 f_t(X_m(t)) \cdot \partial_t X_m(t)\right|.$$

Thus,

$$\left|\partial_t \log |Df_t^i(x(t))|\right| = \left|\sum_{m=0}^{i-1} \frac{\partial_t Df_t(X_m(t))}{Df_t(X_m(t))}\right| \le \sum_{m=0}^{i-1} \left(C_2 + \frac{C_3 |\partial_t X_m(t)|}{\mathrm{dist}(X_m(t), \mathcal{C})}\right) \le \lambda^2,$$

provided that N is large enough. Since

$$\log \frac{Df_t^i(x(t))}{Df^i(x)} = \int_0^t \partial_t \log |Df_t^i(x(t))|,$$

the inequality (4.50) holds. $\square$

In the following, let K be the constant given by Lemma 4.32. Furthermore, if x is a stable point, we always use the notation $x(\cdot)$ to denote the function which is given by Definition 4.7

Definition 4.8. Given $N > 0$ and an admissible family $\mathscr{P} = \{\mathcal{P}(t)\}_t$, we say that an interval $I = [a, b]$ is *a stable branch of order N with respect to $\mathscr{P}$*, if the following hold:

(1) $f^N(a)$ and $f^N(b)$ are adjacent points in $\mathcal{P}(0)$;

(2) for each $x \in I$, x is a N^K-bounded N^{-4K}-stable point of order N with respect to $\mathscr{P}$;

(3) for each $t \in [-N^{-4K}, N^{-4K}]$, $f_t^N : [a(t), b(t)] \to [0, 1]$ is a diffeomorphism onto its image and

$$e^{-1}|Df_t^j(x)| \leq |Df_t^j(y)| \leq e|Df_t^j(x)|$$

holds for any $x, y \in [a(t), b(t)]$ and $0 \leq j \leq N$.

Write

$$S_{k,t}(x) = \frac{1}{k}\sum_{i=0}^{k-1} \phi(f_t^i(x)),$$

then we have the following result.

Lemma 4.33. *There exists $\zeta > 0$ such that for each admissible family $\mathscr{P}$ of size less than ζ, the following holds provided that N is large enough. If $I = [a, b]$ is a stable branch of order N with respect to $\mathscr{P}$, then we have*

(i) for any $t \in [-N^{-4K}, N^{-4K}]$, $x \in [a, b]$, $y \in [a(t), b(t)]$ and $k \leq N$,

$$|S_{k,0}(x) - S_{k,t}(y)| < \delta/4; \tag{4.57}$$

(ii) for any $t \in [-N^{-4K}, N^{4K}]$,

$$\frac{|a(t)-b(t)|}{|a-b|} \geq 1 - \frac{1}{N^4}. \tag{4.58}$$

Proof. (i) By uniform continuity of ϕ, let $\zeta > 0$ be a small constant such that $|\phi(x) - \phi(y)| < \delta/4$ holds for any $|x-y| \leq 3\zeta$. To prove (4.57), it suffices to prove that $|f^i(x) - f_t^i(y)| \leq 3\zeta$ holds for any $i = 0, 1, \ldots, N-1$. By the definition of the stable branch, we obtain that $f_t^{N-i} : [f_t^i(a(t)), f_t^i(b(t))] \to [0,1]$ has a bounded distortion for any $t \in [-N^{-4K}, N^{-4K}]$. Note that $\mathcal{P}(t)$ is a f_t-forward invariant set. Hence, we have $|f_t^i(a(t)) - f_t^i(b(t))| \leq \|\mathscr{P}\| \leq \zeta$ holds for any $t \in [-N^{-4K}, N^{-4K}]$ and $0 \leq i \leq N-1$. Furthermore, we have that $|f_t^i(a(t)) - f^i(a(0))| < N^{-3K}$. Thus, we get that $|f^i(x) - f_t^i(y)| \leq 2\zeta + N^{-3K} \leq 3\zeta$, provided that N is large enough. The statement (i) follows.

(ii) Denote $P = f^N(a)$ and $Q = f^N(b)$. For any $x \in [a,b]$, we have that

$$f_t^N(x(t)) = \frac{P(t) - Q(t)}{P(0) - Q(0)} \cdot (f^N(x) - P(0)) + P(t).$$

For any $t \in [-N^{-4K}, N^{-4K}]$, by the mean value theorem, there exists $x_0 \in [a,b]$ such that

$$\frac{|a(t)-b(t)|}{|a-b|} = \frac{|P(t)-Q(t)|}{|P(0)-Q(0)|} \cdot \frac{|Df^N(x_0)|}{|Df_t^N(x_0(t))|}.$$

Since $P(t), Q(t) \in \mathcal{P}(t)$, then there exists $C_0 > 0$ depending only on $\mathscr{P}$ such that

$$\frac{|P(t)-Q(t)|}{|P(0)-Q(0)|} \geq 1 - C_0 N^{-4K}.$$

Note that x_0 is a N^K-bounded N^{-4K}-stable point with respect to $\mathscr{P}$. By (4.50), the statement (ii) follows. □

4.3.3 A family of stable branches

Given $N > 0$, we define

$$G_N = \left\{ x \in [0,1] : \left| S_{N,0}(x) - \int \phi d\mu_0 \right| \geq \frac{\delta}{4} \right\}.$$

For each closed interval I, each $N > 0$ and an admissible family $\mathscr{P}$, let $\mathcal{M}_N(I, \mathscr{P})$ be the collection of closed intervals $J \subset I$ with the following properties:

- $J \setminus G_N \neq \emptyset$ and $\mathrm{dist}(J, \partial I) > N^{-3}$;
- there exists $N_J \in [N, N^2]$ such that J is a stable branch of order N_J with respect to $\mathscr{P}$.

Lemma 4.34. *There exists $\zeta > 0$ such that for each admissible family $\mathscr{P}$ of size less than ζ, the following holds provided that N is large enough. For each closed interval I, there exists a finite family $\{I_i\} \subset \mathcal{M}_N(I, \mathscr{P})$ such that*

$$\sum_i |I_i| > |I| - \frac{6}{N^3}$$

and such that for each $i \neq j$, the interiors of I_i and I_j are disjoint.

Proof. By the definition of the normalized regular family, there exists a constant $C > 0$ such that for any $x_1, x_2 \in [0,1]$ and $t \in [-1,1]$ with $|x_1 - x_2|$ small and

$$(2e)^{-2}|Df(x_1)| \leq |Df_t(x_2)| \leq (2e)^2|Df(x_1)|,$$

we have that $C^{-1}\mathrm{dist}(x_1, \mathcal{C}) \leq \mathrm{dist}(x_2, \mathcal{C}) \leq C\mathrm{dist}(x_1, \mathcal{C})$. Fix $\tau > 0$ and $\varepsilon > 0$ small. Let $\kappa \in (0,1)$ and $\eta > 0$ be the constants such that the conclusions of Proposition 4.2 hold for these constants τ and ε. Moreover, fix $\zeta > 0$ small such that $\zeta < \eta\theta_0 C^{-1}(2e)^{-3}$, where θ_0 is the constant given by Lemma 3.10.

Fix an admissible family $\mathscr{P} = \{\mathcal{P}(t)\}_t$ of size less than ζ and a closed interval I. For each N large enough, let $N_0 = [N/\kappa] + 1$ and let $F_N(I) \subset I \setminus G_N$ be the collection of points x such that

$$\mathrm{dist}(x, \partial I) > \frac{2}{N^3}, \sum_{i=0}^{N_0-1} q_\varepsilon(f^i(x)) < \tau N_0 \text{ and } \max_{i=0}^{N_0-1} q_\varepsilon(f^i(x)) < 5\log N_0.$$

Since $0 \in \Lambda_\Theta(\phi, \delta)$, then $|G_N| \leq \Theta e^{-N/\Theta}$. Combining this with Proposition 4.26 and $N < N_0$, we can obtain that

$$|I \setminus F_N(I)| \leq \Theta e^{-N/\Theta} + \frac{4}{N^3} + e^{-\frac{\tau N}{8}} + \frac{1}{N^3} \leq \frac{6}{N^3},$$

provided that N is large enough.

For any $x \in F_N(I)$, by Proposition 4.2, there exists $N_x \in [\kappa N_0, N_0]$ such that N_x is η-hyperbolic time for the point x under the iteration of f. By Lemma 3.10, for any $y \in J_x$ and $0 \le i \le N_x$, we have that

$$e^{-1}|Df^i(y)| \le |Df^i(x)| \le e|Df^i(y)|, \tag{4.59}$$

where

$$J_x = \left[x - \frac{\theta_0}{\mathrm{A}(x, f, N_x)}, x + \frac{\theta_0}{\mathrm{A}(x, f, N_x)}\right].$$

Hence, we have that $|Df^{N_x}(J_x)| \asymp |Df^{N_x}(x)| \cdot |J_x|$. Since $|Df^{N_x}(x)|$ is exponentially large with N_x, then we have that $\mathrm{dist}(\partial J_x, \partial I) > N^{-3}$ provided that N is large enough.

We then show that there exists a closed interval I_x containing x with $I_x \subset J_x$ such that I_x is a stable branch of order N_x with respect to $\mathscr{P}$. By the bounded distortion property of $f^{N_x}|J_x$, we have that $|f^{N_x}(J_x)| \ge 2\eta\theta_0 e^{-1} > 2\zeta$. Hence, there exists a closed interval $I_x = [a, b] \subset J_x$ containing x such that $f^{N_x}(a)$ and $f^{N_x}(b)$ are the adjacent points in $\mathcal{P}(0)$. By (4.59), for each $y \in I_x$, we obtain that

$$e^{-2}|Df(f^i(y))| \le |Df(f^i(x))| \le e^2|Df(f^i(y))|$$

holds for $0 \le i < N_x$. By the local behave of f near critical points, for each $y \in I_x$, we have

$$\max_{i=0}^{N_x-1} q_\varepsilon(f^i(y)) < 10 \log N_x,$$

provided that N is large enough. By Lemma 4.32, y is a N_x^K-bounded N_x^{-4K}-stable point of order N_x with respect to $\mathscr{P}$. Thus, I_x satisfies the properties (1) and (2) in the Definition 4.8.

To show I_x satisfies the property (3), by Lemma 3.10, it suffices to show that

$$J(t) := \left[a(t), a(t) + \frac{\theta_0}{\mathrm{A}(a(t), f_t, N_x)}\right] \supset [a(t), b(t)] \tag{4.60}$$

holds for any $|t| \le N_x^{-4K}$. Let us prove (4.60). By Lemma 4.32 and (4.59), for any $|t| \le N_x^{-4K}$ and $0 \le i \le N_x$, we have that

$$\frac{1}{2e} \le \frac{|Df_t^i(a(t))|}{|Df^i(x)|} \le 2e,$$

which implies that

$$\frac{|Df_t^{N_x}(a(t))|}{\mathrm{A}(a(t), f_t, N_x)} \geq \frac{1}{4e^2C} \cdot \frac{|Df^{N_x}(x)|}{\mathrm{A}(x, f, N_x)} > \frac{2e\zeta}{\theta_0}.$$

Hence, $|f_t^{N_x}(J(t))| \geq 2\zeta$. Since $|f_t^{N_x}([a(t), b(t)])| \leq \|\mathscr{P}\| \leq \zeta$, it follows (4.60). Thus, $I_x \in \mathcal{M}_N(I, \mathscr{P})$.

For any $x, y \in F_N(I)$, let I_x and I_y be defined as above. If $\mathrm{int}(I_x) \cap \mathrm{int}(I_y) \neq \emptyset$, then $I_x \subset I_y$ or $I_x \subset I_y$. Otherwise, assume $N_x \geq N_y$, then $\partial I_y \cap \mathrm{int}(I_x) \neq \emptyset$. Let $z \in \partial I_y \cap \mathrm{int}(I_x)$, then $f^{N_x}(z) = f^{N_x - N_y}(f^{N_y}(z)) \in \mathcal{P}(0)$, which implies that I_x can not be the stable branch of order N_x with respect to $\mathscr{P}$. Hence, $I_x \subset I_y$ or $I_y \subset I_x$. Besides, the boundary points of each I_x are contained in $f^{-N}(\mathcal{P}(0))$ which is a finite set. Hence, there exists a finite family $\{I_i\}_i$ of closed subintervals of I such that the interior of I_i and I_j are disjoint for any $i \neq j$. Furthermore, we can obtain that

$$\sum_i |I_i| \geq |F_N(I)| \geq |I| - \frac{6}{N^3}.$$

The statement follows. □

4.3.4 Proof of Reduced Theorem B*

We shall use $\xi_n^{(z)} = f_t^{n+1}(z(t))$ to estimate measure of sets of bad parameters. Since $0 \in \Lambda_\Theta(\phi, \delta)$, then $z(\cdot)$ satisfies the property $(*)$ at the parameter 0. Let λ_1, λ_2, η, κ and r_n be given in the definition of the property $(*)$ for $z(\cdot)$ at the parameter 0.

To prove the Reduced Theorem B*, we shall first define preferred pair for each n large as follows. Let $C_0 = \max_{[0,1]} |\phi|$ and let $\rho = \min\{\delta(8C_0)^{-1}, 4^{-1}\}$, we say that a pair (m, N) is *a preferred pair for* n, if the following hold:

- $m + 1 + N = n$, $m + 1 \leq \rho n$ and $2^{-1}\kappa\rho N \leq m$;
- $|\xi_m^{(z)}([-r_m, r_m])| \geq \eta$.

For any n large enough, a preferred pair (m, N) for n exists. Indeed, there exists an integer m_0 such that $2^{-1}\rho n \leq m_0 \leq \rho n - 1$. By the property $(*)$ for z at the parameter 0, there exists $m \in [\kappa m_0, m_0]$ such that $|\xi_m^{(z)}([-r_m, r_m])| \geq \eta$. Then let $N = n - m - 1$ and this pair (m, N) is a preferred pair for n.

If (m, N) is a preferred pair for n, then $r_m < N^{-8K}$ holds provided that n is large enough. Hence, given a closed interval I, an admissible family $\mathscr{P}$ and $J \in \mathcal{M}_N(I, \mathscr{P})$, for any $x \in J$, x is a N^K-bounded r_m-stable point of order N. Then we have the following result.

Lemma 4.35. *For any admissible family $\mathscr{P}$, the following holds provided that n is large enough. Let (m, N) be a preferred pair for n and $I = \xi_m^{(z)}([-r_m, r_m])$. For any $[x_1, x_2] \in \mathcal{M}_N(I, \mathscr{P})$, there exists a closed interval $T \subset [-r_m, r_m]$ with the following properties:*

- *for any $t \in T$, $\xi_m^{(z)}(t) \in [x_1(t), x_2(t)]$;*
- $|\xi_m^{(z)}(T)| \geq (1 - N^{-3}) \cdot |x_1 - x_2|$.

Proof. By the definition of $\mathcal{M}_N(I, \mathscr{P})$, for any $t \in [-r_m, r_m]$ and $i = 1, 2$, we have that $|x_i(t) - x_i(0)| \leq N^{2K} \cdot r_m \leq N^{2K} \cdot \lambda_1^{-m} \leq N^{-3}$, provided that N is large enough. Hence, $x_i(t) \in I$ holds for any $t \in [-r_m, r_m]$. This implies that for each $i = 1, 2$, there exists t_i such that $\xi_m^{(z)}(t_i) = x_i(t_i)$. Let T be the closed interval bounded by t_1 and t_2. We shall prove the statement for this T.

By the bounded distortion property of $\xi_m^{(z)}|_{[-r_m, r_m]}$, we have that

$$\begin{aligned} |x_1(t_1) - x_2(t_2)| &= |\xi_m^{(z)}(t_1) - \xi_m^{(z)}(t_2)| \\ &\geq \frac{\lambda_2^m}{e}|t_1 - t_2| \geq N^{2K}|t_1 - t_2| \geq |x_2(t_1) - x_2(t_2)|, \end{aligned} \tag{4.61}$$

provided that n is large enough. Together with $x_1(t_1) < x_2(t_1)$, this implies that $x_1(t_1) < x_2(t_2)$. Then we obtain that $\xi_m^{(z)}(t) \in [x_1(t_1), x_2(t_2)]$ holds for any $t \in T$. Similarly as above, we have that $|x_1(t_1) - \xi_m^{(z)}(t)| \geq |x_1(t_1) - x_1(t)|$. Hence, $\xi_m^{(z)}(t) \geq x_1(t)$. With the same process, we can get $\xi_m^{(z)}(t) \leq x_2(t)$. Thus, $\xi_m^{(z)}(t) \in [x_1(t), x_2(t)]$ holds for any $t \in T$. This proves the first assertion of the lemma.

To prove the second assertion, note that

$$\frac{|\xi_m^{(z)}(t_1) - \xi_m^{(z)}(t_2)|}{|x_2(t_1) - x_2(t_2)|} \geq \frac{1}{e} \cdot \frac{\lambda_2^m}{N^{2K}} \cdot \frac{|t_1 - t_2|}{|t_1 - t_2|}.$$

Furthermore, by definition, we have $|\xi_m^{(z)}(t_1) - \xi_m^{(z)}(t_2)| = |x_1(t_1) - x_2(t_2)|$. Combined with part (ii) of Lemma 4.33, we obtain that

$$|\xi_m^{(z)}(T)| \geq (1 - N^{-3}) \cdot |x_1 - x_2|.$$

$\square$

Proof of Reduced Theorem B.* Let $\zeta > 0$ be a small constant such that the conclusions of Lemmas 4.33 and 4.34 hold. By Proposition 4.30 and Lemma 4.31, there exist $\omega > 0$ such that an ω-admissible family $\mathscr{P}$ with size less than ζ exists and such that for any $t \in [-\omega, \omega] \cap \Lambda_\Theta(\phi, \delta)$, we have

$$\left| \int \phi d\mu_0 - \int \phi d\mu_t \right| < \frac{\delta}{4}. \tag{4.62}$$

For each n large enough, let (m, N) be a preferred pair for n and let $I = \xi_m^{(z)}([-r_m, r_m])$. Consider an interval $[x_1, x_2] \in \mathcal{M}_N(I, \mathscr{P})$. Let T be the closed interval given by Lemma 4.35 for the interval $[x_1, x_2]$. We shall first show that if $t \in \Lambda_\Theta(\phi, \delta) \cap T$, then $t \notin E_n(\phi, \delta)$. Fix $t \in \Lambda_\Theta(\phi, \delta) \cap T$. By the definition of $\mathcal{M}_N(I, \mathscr{P})$, there exists a point $x \in [x_1, x_2]$ such that

$$\left| S_{N,0}(x) - \int \phi d\mu_0 \right| < \frac{\delta}{4}.$$

Combining this, Lemma 4.35, (4.57) and (4.62), we obtain that

$$\left| S_{N,t}(\xi_m^{(z)}(t)) - \int \phi d\mu_t \right| < \frac{3\delta}{4}. \tag{4.63}$$

Note that

$$\left(\frac{m+1}{n} \right) \cdot \left| S_{m+1,t}(z(t)) - \int \phi d\mu_t \right| < \frac{\delta}{4}.$$

Hence, $t \notin E_n(\phi, \delta)$.

Let $\{I_i\}_i \subset \mathcal{M}_N(I, \mathscr{P})$ be the finite family given by Lemma 4.34. For each i, let $T_i \subset [-r_m, r_m]$ be the closed interval given by Lemma 4.35 for I_i. By Lemmas 4.34 and 4.35, we have that the interiors of T_i and T_j are disjoint for each $i \neq j$. Moreover,

$$\sum_i \xi_m^{(z)}(T_i) > \left(1 - \frac{1}{N^3}\right) \sum_i |I_i| > \left(1 - \frac{1}{N^3}\right) \cdot \left(|I| - \frac{6}{N^3}\right).$$

By the bounded distortion property of $\xi_m^{(z)}|_{[-r_m, r_m]}$ and $|I| \geq \eta$, we obtain that

$$\frac{|[-r_m, r_m] \setminus \bigcup_i T_i|}{2r_m} < \frac{1}{n^2},$$

provided that n is large enough. By the above discussion, we have

$$[-r_m, r_m] \cap \Lambda_\Theta(\phi, \delta) \cap E_n(\phi, \delta) \subset [-r_m, r_m] \setminus \bigcup_i T_i.$$

The Reduced Theorem B* is proved. $\square$

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